On the existence of FIR zero-forcing equalizers for nonredundantly-precoded transmissions through FIR channels

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***Abstract*— We consider in this letter the problem of perfectly equalizing finite-impulse response (FIR) channels in digital communication systems employing nonredundant precoding at the transmitter. In this case, as it is well-known, there is no linear FIR zero-forcing (ZF) block equalizer that is capable of perfectly recovering information symbols in the absence of noise. We show that, when the symbol constellation is improper, it is possible to design widely-linear FIR-ZF block equalizers which, unlike their linear counterparts, process both the received signal and its complex-conjugate version. In addition, we propose a simple procedure to synthesize universal modulation-induced cyclostationary precoders that assures perfect symbol recovery, regardless of the channel-zero locations.**

***Index Terms*— Improper processes, nonredundant precoding, block equalization, widely-linear filtering.**

I. INTRODUCTION AND PRELIMINARIES

LET $s(n) \in \mathbb{C}$ be a zero-mean sequence of independent and identically distributed (i.i.d.) information symbols, with second-order moments $\sigma_s^2 \triangleq \mathrm{E}[|s(n)|^2]$ and $\gamma_s \triangleq \mathrm{E}[s^2(n)]$, and symbol period T_s. In linearly precoded block communications [1], a block of N consecutive symbols $\boldsymbol{s}(k) \triangleq [s(k\,N), s(k\,N+1), \ldots, s(k\,N+N-1)]^T \in \mathbb{C}^N$, with $(\cdot)^T$ denoting the transpose, is subject to a linear transformation $\overline{\boldsymbol{s}}(k) \triangleq [\overline{s}(k\,P), \overline{s}(k\,P+1), \ldots, \overline{s}(k\,P+P-1)]^T = \boldsymbol{\mathcal{F}}\,\boldsymbol{s}(k) \in \mathbb{C}^P$, where $P \geq N$ and $\boldsymbol{\mathcal{F}} \in \mathbb{C}^{P \times N}$ is a full-column rank precoding matrix. The encoded sequence $\overline{s}(n)$ is transmitted *in lieu* of $s(n)$ and, thus, the complex envelope of the received signal, after filtering and baud-rate sampling, is

$$r(n) = \sum_{i=0}^{L_c} c(i)\,\overline{s}(n-i) + w(n)\,, \qquad (1)$$

where $c(n) \in \mathbb{C}$ denotes the overall impulse response of the linear time-invariant discrete-time signal channel, which is modeled as a causal finite-impulse response (FIR) filter of order $0 < L_c < P$, with $c(0), c(L_c) \neq 0$, whereas $w(n) \in \mathbb{C}$ represents the additive stationary noise at the output of the receiving filter. By introducing the data vector $\boldsymbol{r}(k) \triangleq [r(k\,P), r(k\,P+1), \ldots, r(k\,P+P-1)]^T \in \mathbb{C}^P$, and accounting for (1), we obtain the concise vector model

$$\boldsymbol{r}(k) = \boldsymbol{C}_0\,\boldsymbol{\mathcal{F}}\,\boldsymbol{s}(k) + \boldsymbol{C}_1\,\boldsymbol{\mathcal{F}}\,\boldsymbol{s}(k-1) + \boldsymbol{w}(k)\,, \qquad (2)$$

where $\boldsymbol{C}_0 \in \mathbb{C}^{P \times P}$ and $\boldsymbol{C}_1 \in \mathbb{C}^{P \times P}$ are Toeplitz lower- and upper-triangular matrices, whose first column and row are given by $[c(0), c(1), \ldots, c(L_c), 0, \ldots, 0]^T$ and $[0, \ldots, 0, c(L_c), c(L_c-1), \ldots, c(1)]$, respectively, whereas $\boldsymbol{w}(k) \in \mathbb{C}^P$ is defined similarly to $\boldsymbol{r}(k)$.

Sufficient conditions have been derived in [1] to guarantee that, with *redundant* linear precoders (i.e., $P > N$), FIR channels are perfectly equalized in the absence of noise, by resorting to linear (L) FIR zero-forcing (ZF) block equalizers, regardless of the locations of the channel zeros. On the other hand, *nonredundant* linear precoders (i.e., $P = N$), based on modulation-induced cyclostationarity (MIC) [2], [3], exhibit the attractive feature that they introduce *spectral redundancy* in the received signal without decreasing the information rate. It was shown [2]–[5] that, for MIC-based communication techniques, it is possible to derive blind estimation methods for channel identification, synchronization, and equalization, which are resilient to the location of the channel zeros, power spectral density of noise, and channel order mismatches. However, it has been recognized [3], [4] that, in comparison with redundant precoding techniques, the drawback of MIC-based approaches is the lack of L-FIR-ZF equalizers for FIR channels, which implies that L-FIR minimum mean-square error (MMSE) equalizer performances exhibit a marked error floor at high signal-to-noise ratio (SNR) values.

In this letter, we focus our attention on nonredundantly-precoded transmissions. First, we show that, for an arbitrary nonsingular precoding matrix $\boldsymbol{\mathcal{F}} \in \mathbb{C}^{N \times N}$, FIR-ZF block equalization is possible by additionally exploiting the *improper* [6] nature of many symbol constellations. Exploitation of such a property leads to the synthesis of *widely-linear* (WL) [7] block equalizers, which jointly elaborate $\boldsymbol{r}(k)$ and its conjugate version $\boldsymbol{r}^*(k)$. Secondly, since WL-FIR-ZF solutions are available, it is also shown experimentally that the WL-FIR-MMSE equalizer exhibits satisfactory performances, significantly outperforming its linear counterpart. Finally, focusing our attention on MIC-based communication techniques, we develop a systematic algorithm for synthesizing *universal* precoders that guarantee WL-FIR-ZF block equalization for *any* FIR channel of order L_c, without requiring channel state information (CSI) at the transmitter.

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II. WIDELY-LINEAR FIR-ZF BLOCK EQUALIZATION

All the real-valued symbol sequences $s(n)$, such as BPSK, DBPSK, M-ASK, and many conjugate symmetric complex-valued symbol constellations, such as OQPSK, OQAM, and binary CPM, MSK, GMSK, fall into the class of improper [6] random processes, since they exhibit a nonvanishing conjugate second-order moment, i.e., $\gamma_s = \mathrm{E}[s^2(n)] \neq 0$. In all these cases, the improper nature of $s(n)$ is a consequence of the linear dependence existing between $s(n)$ and its conjugate version $s^*(n)$, i.e., $s^*(n) = e^{j\,2\pi\delta n}\, s(n)$ for any $n \in \mathbb{Z}$ and for any realization of the random process $s(n)$. Observe that real modulation schemes fulfill the previous relation with $\delta = 0$, i.e., $s^*(n) = s(n)$, whereas for complex modulation formats, such as OQPSK, OQAM, and MSK-type, the above-mentioned relation is satisfied [8], [9] with $\delta = 1/2$, i.e., $s^*(n) = (-1)^n\, s(n)$. The key observation that motivates our approach is that, for the considered modulation schemes, the linear dependence existing between $s(n)$ and $s^*(n)$ is an "intrinsic" redundancy contained in the original symbol sequence $s(n)$, which can be suitably exploited to synthesize FIR-ZF equalizers even with $P = N$. To this end, we observe that such a redundancy can be exploited by resorting to WL [7] processing[1] of $\boldsymbol{r}(k)$, that is, $\boldsymbol{y}(k) = \boldsymbol{G}_1\, \boldsymbol{r}(k) + \boldsymbol{G}_2\, \boldsymbol{r}^*(k)$. In the absence of noise, accounting for (2) and assuming that N is an even number[2], the input-output relation of a WL-FIR block equalizer can be explicitly written as

$$\boldsymbol{y}(k) = \underbrace{\left[\ \boldsymbol{G}_1, \boldsymbol{G}_2\ \right]}_{\widetilde{\boldsymbol{\mathcal{G}}} \in \mathbb{C}^{N \times 2N}} \underbrace{\begin{bmatrix} \boldsymbol{C}_0\, \boldsymbol{\mathcal{F}} \\ \boldsymbol{C}_0^*\, \boldsymbol{\mathcal{F}}^* \boldsymbol{J} \end{bmatrix}}_{\widetilde{\boldsymbol{\mathcal{H}}}_0 \in \mathbb{C}^{2N \times N}} \boldsymbol{s}(k) + \left[\ \boldsymbol{G}_1, \boldsymbol{G}_2\ \right] \underbrace{\begin{bmatrix} \boldsymbol{C}_1\, \boldsymbol{\mathcal{F}} \\ \boldsymbol{C}_1^*\, \boldsymbol{\mathcal{F}}^* \boldsymbol{J} \end{bmatrix}}_{\widetilde{\boldsymbol{\mathcal{H}}}_1 \in \mathbb{C}^{2N \times N}} \boldsymbol{s}(k-1)\,, \quad (3)$$

where $\boldsymbol{J} \triangleq \mathrm{diag}[1, e^{j\,2\pi\delta}, \ldots, e^{j\,2\pi\delta(N-1)}] \in \mathbb{C}^{N \times N}$ and, hereinafter, we assume that $\boldsymbol{\mathcal{F}} \in \mathbb{C}^{N \times N}$ is an arbitrary nonsingular matrix. Thus, from (3), the WL-ZF condition $\boldsymbol{y}(k) = \boldsymbol{s}(k)$ is equivalent to the following ones: $\widetilde{\boldsymbol{\mathcal{G}}}\, \widetilde{\boldsymbol{\mathcal{H}}}_0 = \mathbf{I}_N$ and $\widetilde{\boldsymbol{\mathcal{G}}}\, \widetilde{\boldsymbol{\mathcal{H}}}_1 = \mathbf{O}_{N \times N}$, where $\mathbf{I}_N$ and $\mathbf{O}_{N\times N}$ denote the $N \times N$ identity matrix and the $N \times N$ zero matrix, respectively.

Let us first consider the second ZF condition: $\widetilde{\boldsymbol{\mathcal{G}}}\, \widetilde{\boldsymbol{\mathcal{H}}}_1 = \boldsymbol{G}_1\, \boldsymbol{C}_1\, \boldsymbol{\mathcal{F}} + \boldsymbol{G}_2\, \boldsymbol{C}_1^*\, \boldsymbol{\mathcal{F}}^* \boldsymbol{J} = \mathbf{O}_{N \times N}$, which assures complete elimination of the *interblock interference* (IBI). Owing to the structure of $\boldsymbol{C}_1$ and partitioning accordingly $\boldsymbol{G}_1 = [\boldsymbol{G}_{1,1}, \boldsymbol{G}_{1,2}]$ and $\boldsymbol{\mathcal{F}} = [\boldsymbol{\mathcal{F}}_1^T, \boldsymbol{\mathcal{F}}_2^T]^T$, we obtain $\boldsymbol{G}_1\, \boldsymbol{C}_1\, \boldsymbol{\mathcal{F}} = \boldsymbol{G}_{1,1}\, \overline{\boldsymbol{C}}_1\, \boldsymbol{\mathcal{F}}_2$, where $\boldsymbol{G}_{1,1} \in \mathbb{C}^{N \times L_c}$, $\boldsymbol{G}_{1,2} \in \mathbb{C}^{N \times (N-L_c)}$, $\boldsymbol{\mathcal{F}}_1 \in \mathbb{C}^{(N-L_c) \times N}$, $\boldsymbol{\mathcal{F}}_2 \in \mathbb{C}^{L_c \times N}$, and $\overline{\boldsymbol{C}}_1 \in \mathbb{C}^{L_c \times L_c}$ is a Toeplitz upper-triangular matrix, whose first row is given by $[c(L_c), c(L_c-1), \ldots, c(1)]$. Similarly, by partitioning $\boldsymbol{G}_2 = [\boldsymbol{G}_{2,1}, \boldsymbol{G}_{2,2}]$, with $\boldsymbol{G}_{2,1} \in \mathbb{C}^{N \times L_c}$, $\boldsymbol{G}_{2,2} \in \mathbb{C}^{N \times (N-L_c)}$, it follows that $\boldsymbol{G}_2\, \boldsymbol{C}_1^*\, \boldsymbol{\mathcal{F}}^* \boldsymbol{J} = \boldsymbol{G}_{2,1}\, \overline{\boldsymbol{C}}_1^{\,*}\, \boldsymbol{\mathcal{F}}_2^*\, \boldsymbol{J}$. Hence, the IBI-free condition becomes $[\boldsymbol{G}_{1,1}, \boldsymbol{G}_{2,1}]\, \mathrm{diag}[\overline{\boldsymbol{C}}_1, \overline{\boldsymbol{C}}_1^{\,*}]\, [\boldsymbol{\mathcal{F}}_2^T, (\boldsymbol{\mathcal{F}}_2^* \boldsymbol{J})^T]^T = \mathbf{O}_{N \times N}$, where, since $c(L_c) \neq 0$, the block diagonal matrix $\mathrm{diag}[\overline{\boldsymbol{C}}_1, \overline{\boldsymbol{C}}_1^{\,*}]$ is nonsingular, i.e., $\mathrm{rank}(\mathrm{diag}[\overline{\boldsymbol{C}}_1, \overline{\boldsymbol{C}}_1^{\,*}]) = \mathrm{rank}(\overline{\boldsymbol{C}}_1) + \mathrm{rank}(\overline{\boldsymbol{C}}_1^{\,*}) = 2\,\mathrm{rank}(\overline{\boldsymbol{C}}_1) = 2\,L_c$. By imposing the constraint that IBI suppression must hold for any pair $(\boldsymbol{\mathcal{F}}_2, \boldsymbol{J})$, one has $[\boldsymbol{G}_{1,1}, \boldsymbol{G}_{2,1}]\, \mathrm{diag}[\overline{\boldsymbol{C}}_1, \overline{\boldsymbol{C}}_1^{\,*}] = \mathbf{O}_{N \times 2L_C}$, which admits the solution $\boldsymbol{G}_{1,1} = \boldsymbol{G}_{2,1} = \mathbf{O}_{N \times L_c}$. Thus, the *general* form of an IBI-free WL-FIR equalizer is $\boldsymbol{G}_1 = \boldsymbol{G}_{1,2}\, \boldsymbol{\Omega}$ and $\boldsymbol{G}_2 = \boldsymbol{G}_{2,2}\, \boldsymbol{\Omega}$, where $\boldsymbol{\Omega} \triangleq [\mathbf{O}_{(N-L_c)\times L_c}, \mathbf{I}_{N-L_c}] \in \mathbb{R}^{(N-L_c)\times N}$.

At this point, we consider the first ZF condition: $\widetilde{\boldsymbol{\mathcal{G}}}\, \widetilde{\boldsymbol{\mathcal{H}}}_0 = \boldsymbol{G}_1\, \boldsymbol{C}_0\, \boldsymbol{\mathcal{F}} + \boldsymbol{G}_2\, \boldsymbol{C}_0^*\, \boldsymbol{\mathcal{F}}^* \boldsymbol{J} = \mathbf{I}_N$. Accounting for the IBI-free structure of the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, we get

$$\widetilde{\boldsymbol{\mathcal{G}}}\, \widetilde{\boldsymbol{\mathcal{H}}}_0 = \underbrace{\left[\ \boldsymbol{G}_{1,2}, \boldsymbol{G}_{2,2}\ \right]}_{\boldsymbol{\mathcal{G}} \in \mathbb{C}^{N \times 2(N-L_c)}} \underbrace{\begin{bmatrix} \overline{\boldsymbol{C}}_0\, \boldsymbol{\mathcal{F}} \\ \overline{\boldsymbol{C}}_0^{\,*}\, \boldsymbol{\mathcal{F}}^* \boldsymbol{J} \end{bmatrix}}_{\boldsymbol{\mathcal{H}}_0 \in \mathbb{C}^{2(N-L_c) \times N}} = \mathbf{I}_N\,, \quad (4)$$

where $\overline{\boldsymbol{C}}_0 \triangleq \boldsymbol{\Omega}\, \boldsymbol{C}_0 \in \mathbb{C}^{(N-L_c)\times N}$ is a Toeplitz upper-triangular matrix, whose first row is given by $[c(L_c), c(L_c-1), \ldots, c(0), 0, \ldots, 0]$. Matrix equation (4) admits solutions if and only if (iff) $\boldsymbol{\mathcal{H}}_0$ is tall and full-column rank, i.e., **A1)** $2\,(N - L_c) \geq N$ and **A2)** $\mathrm{rank}(\boldsymbol{\mathcal{H}}_0) = N$. In the following, we will assume that $N \geq 2\,L_c$ so as to satisfy condition A1. As to condition A2, we observe that $\mathrm{rank}(\boldsymbol{\mathcal{H}}_0) = N$ iff the null spaces of $\overline{\boldsymbol{C}}_0\, \boldsymbol{\mathcal{F}}$ and $\overline{\boldsymbol{C}}_0^{\,*}\, \boldsymbol{\mathcal{F}}^* \boldsymbol{J}$ intersect only trivially, i.e.,

$$\mathcal{N}(\overline{\boldsymbol{C}}_0\, \boldsymbol{\mathcal{F}}) \cap \mathcal{N}(\overline{\boldsymbol{C}}_0^{\,*}\, \boldsymbol{\mathcal{F}}^* \boldsymbol{J}) = \{\mathbf{0}_N\}\,. \quad (5)$$

Accounting for nonsingularity of $\boldsymbol{\mathcal{F}}$, it follows that an arbitrary vector $\boldsymbol{x} \in \mathbb{C}^N$ belongs to $\mathcal{N}(\overline{\boldsymbol{C}}_0\, \boldsymbol{\mathcal{F}})$ iff the vector $\boldsymbol{\mathcal{F}}\, \boldsymbol{x}$ belongs to $\mathcal{N}(\overline{\boldsymbol{C}}_0)$. Observe that, since $c(0), c(L_c) \neq 0$, matrix $\overline{\boldsymbol{C}}_0$ is full row-rank and, hence, its null space $\mathcal{N}(\overline{\boldsymbol{C}}_0)$ is an L_c-dimensional subspace of $\mathbb{C}^N$. Let $\zeta_1, \zeta_2, \ldots, \zeta_{L_c}$ denote the zeros of the channel transfer function $C(z) \triangleq \sum_{q=0}^{L_c} c(q)\, z^{-q}$. As in [1], for the sake of simplicity, we assume that the zeros of $C(z)$ are distinct: in this case, let $\boldsymbol{v}_q \triangleq [1, \zeta_q, \zeta_q^2, \ldots, \zeta_q^{N-1}]^T \in \mathbb{C}^N$ denote the Vandermonde vector associated with the qth zero of the channel transfer function $C(z)$, the columns of the (rectangular) Vandermonde matrix $\boldsymbol{\mathcal{V}} \triangleq [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{L_c}] \in \mathbb{C}^{N \times L_c}$ constitute [10] a set of linearly independent vectors spanning the subspace[3] $\mathcal{N}(\overline{\boldsymbol{C}}_0)$, i.e., $\mathcal{R}(\boldsymbol{\mathcal{V}}) \equiv \mathcal{N}(\overline{\boldsymbol{C}}_0)$. Consequently, it results that $\boldsymbol{\mathcal{F}}\, \boldsymbol{x}$ belongs to the null space of $\overline{\boldsymbol{C}}_0$ iff there exists a vector $\boldsymbol{\alpha} \in \mathbb{C}^{L_c}$ such that $\boldsymbol{\mathcal{F}}\, \boldsymbol{x} = \boldsymbol{\mathcal{V}}\, \boldsymbol{\alpha}$, which, in its turn, is equivalent to state that $\boldsymbol{x} \in \mathcal{N}(\overline{\boldsymbol{C}}_0\, \boldsymbol{\mathcal{F}})$ iff there exists a vector $\boldsymbol{\alpha} \in \mathbb{C}^{L_c}$ such that $\boldsymbol{x} = \boldsymbol{\mathcal{F}}^{-1} \boldsymbol{\mathcal{V}}\, \boldsymbol{\alpha}$. Similarly, by observing that the zeros of the $\mathcal{Z}$-transform of the conjugate channel coefficients $\{c^*(q)\}_{q=0}^{L_c}$ are given by $\{\zeta_q^*\}_{q=1}^{L_c}$ and that $\boldsymbol{J}$ is a diagonal unitary matrix, i.e., $\boldsymbol{J}^{-1} = \boldsymbol{J}^*$, it can be seen that an arbitrary vector $\boldsymbol{x}' \in \mathbb{C}^N$ belongs to the null space of $\overline{\boldsymbol{C}}_0^{\,*}\, \boldsymbol{\mathcal{F}}^* \boldsymbol{J}$ iff there exists a vector $\boldsymbol{\beta} \in \mathbb{C}^{L_c}$ so that $\boldsymbol{x}' = \boldsymbol{J}^* (\boldsymbol{\mathcal{F}}^*)^{-1} \boldsymbol{\mathcal{V}}^*\, \boldsymbol{\beta}$. Thus, condition (5) holds iff the set $\mathcal{A} \triangleq \{(\boldsymbol{\alpha}, \boldsymbol{\beta})\,|\, \boldsymbol{x} = \boldsymbol{x}'\}$

[1] A WL-FIR block equalizer can jointly elaborate multiple consecutive received blocks; we focus here attention to the case where the WL equalizer elaborates only a single block $\boldsymbol{r}(k)$.

[2] For the considered modulations, one has $\boldsymbol{s}^*(k) = e^{j\,2\pi\delta k N}\, \boldsymbol{J}\, \boldsymbol{s}(k)$. When $s(n)$ is real-valued ($\delta = 0$), it results that $e^{j\,2\pi\delta k N} = 1$; whereas, when $s(n)$ is complex-valued ($\delta = 1/2$), it follows that $e^{j\,2\pi\delta k N} = (-1)^{kN}$. In the latter case, without loss of generality, by assuming that N is even, one has $(-1)^{kN} = 1$. If N is an odd number, a "derotation" [8], [9] of $\boldsymbol{r}^*(k)$ must be performed before evaluating $\boldsymbol{y}(k)$ in (3).

[3] In the case of multiple channel zeros, one must resort to the generalized Vandermonde vectors to describe the null space of $\overline{\boldsymbol{C}}_0$ (see, e.g., [10]).

is trivial, that is, $\mathcal{A} \equiv \{(\mathbf{0}_{L_c}, \mathbf{0}_{L_c})\}$. Note that, after some algebraic manipulations, relation $\boldsymbol{x} = \boldsymbol{x}'$ can be written as a system of equations $\boldsymbol{\mathcal{V}}\,\boldsymbol{\alpha} - \boldsymbol{\Psi}\,\boldsymbol{\mathcal{V}}^*\,\boldsymbol{\beta} = \mathbf{0}_N$, where the matrix $\boldsymbol{\Psi} \triangleq \boldsymbol{\mathcal{F}}\,\boldsymbol{J}^*(\boldsymbol{\mathcal{F}}^*)^{-1} \in \mathbb{C}^{N\times N}$ satisfies the relation $\boldsymbol{\Psi}^*\boldsymbol{\Psi} = \boldsymbol{\Psi}\,\boldsymbol{\Psi}^* = \mathbf{I}_N$. Moreover, observe that, by conjugating this system and, then, by left multiplying the resulting relation by $-\boldsymbol{\Psi}$, one obtains $\boldsymbol{\mathcal{V}}\,\boldsymbol{\beta}^* - \boldsymbol{\Psi}\,\boldsymbol{\mathcal{V}}^*\,\boldsymbol{\alpha}^* = \mathbf{0}_N$. The last two systems of equations show that all the pairs $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{A}$ must exhibit necessarily the following property: $\boldsymbol{\beta} = \boldsymbol{\alpha}^*$. Hence, condition (5) holds iff the system $\boldsymbol{\mathcal{V}}\,\boldsymbol{\alpha} - \boldsymbol{\Psi}\,\boldsymbol{\mathcal{V}}^*\,\boldsymbol{\alpha}^* = \mathbf{0}_N$ admits the unique solution $\boldsymbol{\alpha} = \mathbf{0}_{L_c}$. It can be seen that this happens iff the tall matrix $\boldsymbol{\Phi} \triangleq [\,\boldsymbol{\mathcal{V}}, \boldsymbol{\Psi}\,\boldsymbol{\mathcal{V}}^*\,] \in \mathbb{C}^{N\times 2L_c}$ turns out to be full-column rank, i.e., $\text{rank}(\boldsymbol{\Phi}) = 2\,L_c$. The previous discussion leads to the following Theorem:

Theorem 1: Assuming that $N \geq 2\,L_c$, WL-FIR-ZF equalizers exist iff: **A3)** $\text{rank}(\boldsymbol{\Phi}) = \text{rank}([\,\boldsymbol{\mathcal{V}}, \boldsymbol{\Psi}\,\boldsymbol{\mathcal{V}}^*\,]) = 2\,L_c$. In this case, the general form of a WL-FIR-ZF equalizer is given by $\widetilde{\boldsymbol{\mathcal{G}}}_{\text{zf}} = \boldsymbol{\mathcal{G}}_{\text{zf}}\,\boldsymbol{\Omega}_a$, with $\boldsymbol{\Omega}_a \triangleq \text{diag}[\boldsymbol{\Omega},\boldsymbol{\Omega}] \in \mathbb{R}^{2(N-L_c)\times 2N}$ and $\boldsymbol{\mathcal{G}}_{\text{zf}} = \boldsymbol{\mathcal{H}}_0{}^{\dagger} + \boldsymbol{\mathcal{Y}}\,[\mathbf{I}_{2(N-L_c)} - \boldsymbol{\mathcal{H}}_0\,\boldsymbol{\mathcal{H}}_0{}^{\dagger}]$, where $\boldsymbol{\mathcal{H}}_0{}^{\dagger} \in \mathbb{C}^{N\times 2(N-L_c)}$ is the generalized inverse of $\boldsymbol{\mathcal{H}}_0$ and $\boldsymbol{\mathcal{Y}} \in \mathbb{C}^{N\times 2(N-L_c)}$ is an arbitrary matrix.

Proof: The first part of the theorem has been proven previously. As to the second part, observe that the general structure of an IBI-free equalizer is $\widetilde{\boldsymbol{\mathcal{G}}} = \boldsymbol{\mathcal{G}}\,\boldsymbol{\Omega}_a$. If A3 is satisfied, the general solution of equation $\boldsymbol{\mathcal{G}}\,\boldsymbol{\mathcal{H}}_0 = \mathbf{I}_N$ is given [11] by $\boldsymbol{\mathcal{G}}_{\text{zf}} = \boldsymbol{\mathcal{H}}_0{}^{\dagger} + \boldsymbol{\mathcal{Y}}\,[\mathbf{I}_{2(N-L_c)} - \boldsymbol{\mathcal{H}}_0\,\boldsymbol{\mathcal{H}}_0{}^{\dagger}]$. By substituting solution $\boldsymbol{\mathcal{G}}_{\text{zf}}$ in $\widetilde{\boldsymbol{\mathcal{G}}} = \boldsymbol{\mathcal{G}}\,\boldsymbol{\Omega}_a$, one obtains the general form $\widetilde{\boldsymbol{\mathcal{G}}}_{\text{zf}}$. ■

Remark 1. Observe that, due to the presence of $\boldsymbol{\mathcal{Y}}$ in $\boldsymbol{\mathcal{G}}_{\text{zf}}$, the WL-FIR-ZF equalizer is not unique.

Remark 2. For MIC-based communications, it results that $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_{\text{mic}} \triangleq \text{diag}[f(0), f(1), \ldots, f(N-1)]$, with $f(m) = \rho_m\,e^{j\,\theta_m} \neq 0$, and it can be easily shown that $\boldsymbol{\Psi} = \text{diag}[e^{j\,\psi_0}, e^{j\,\psi_1}, \ldots, e^{j\,\psi_{N-1}}]$, where $\psi_m = 2\,\theta_m - 2\,m\,\pi\,\delta$. Thus, the magnitude of the N-periodic modulating sequence $f(n)$ has no effect on the existence of WL-FIR-ZF solutions. This allows one to choose $|f(n)|$ according to other technical constraints, e.g., so as to preserve the desirable constant-envelope property possibly exhibited by $s(n)$.

Corollary 1: Assume that $f(n)$ is chosen such that the phases $\{\theta_m\}_{m=0}^{N-1}$ (see Remark 2) are *harmonically* related, i.e., $\theta_m = m\,\pi\,\theta$, with $\theta \in [0, 2)$. Then, WL-FIR-ZF equalizers exist iff the channel zeros satisfy the following relation: **A4)** $\zeta_{q_1} \neq \zeta_{q_2}^*\,e^{j\,2\pi(\theta-\delta)}$, for any $q_1, q_2 \in \{1, 2, \ldots, L_c\}$.

Proof: Under the assumption that the phases $\{\theta_m\}_{m=0}^{N-1}$ are harmonically related, it results that $\boldsymbol{\Psi} = \text{diag}[1, e^{j\,2\pi(\theta-\delta)}, (e^{j\,2\pi(\theta-\delta)})^2, \ldots, (e^{j\,2\pi(\theta-\delta)})^{N-1}]$ and, consequently, $\boldsymbol{\Phi}$ turns out to be a Vandermonde matrix whose columns are represented by the Vandermonde vectors associated with the complex numbers $\zeta_1, \zeta_2, \ldots, \zeta_{L_c}, \zeta_1^*\,e^{j\,2\pi(\theta-\delta)}, \zeta_2^*\,e^{j\,2\pi(\theta-\delta)}, \ldots, \zeta_{L_c}^*\,e^{j\,2\pi(\theta-\delta)}$. Therefore, condition A3 is satisfied iff these $2\,L_c$ numbers are all distinct, which, accounting for the assumption that the zeros $\zeta_1, \zeta_2, \ldots, \zeta_{L_c}$ are distinct, leads to relation A4. ■

Remark 3. Let us assume that precoding is not employed, i.e., $f(n) = 1$ for any $n \in \mathbb{Z}$. In this case, the phases $\{\theta_m\}_{m=0}^{N-1}$ are harmonically related, with $\theta = 0$. Hence, by invoking Corollary 1, it follows that WL-FIR-ZF equalizers exist iff $\zeta_{q_1} \neq \zeta_{q_2}^*\,e^{-j\,2\pi\delta}$, for any $q_1, q_2 \in \{1, 2, \ldots, L_c\}$. This result, also derived in [8] in a less general form with reference to WL processing for conventional (i.e., serial) equalization, establishes that WL-FIR-ZF equalization is possible for FIR channel satisfying A4 with $\theta = 0$, without requiring any precoder in transmission and without resorting to oversampling and/or multiple antennas at the receiver.

Corollary 1 evidences that, when $\{\theta_m\}_{m=0}^{N-1}$ are harmonically related, the existence of WL-FIR-ZF equalizers for MIC-based communication systems depends on the channel-zeros configuration. As a matter of fact, it is interesting to investigate whether there exist precoding matrices $\boldsymbol{\mathcal{F}}_{\text{mic}}$ assuring that condition A3 holds regardless of the channel-zero locations (*universal precoding*). Mathematically, this can be rephrased as follows: given $\boldsymbol{J}$, find a matrix $\boldsymbol{\Psi}(\boldsymbol{\psi}) \triangleq \text{diag}[e^{j\,\psi_0}, e^{j\,\psi_1}, \ldots, e^{j\,\psi_{N-1}}]$, where $\boldsymbol{\psi} \triangleq [\psi_0, \psi_1, \ldots, \psi_{N-1}]^T \in \mathbb{C}^N$, with $\psi_m = 2\,\theta_m - 2\,m\,\pi\,\delta$, such that condition A3 holds for *any* Vandermonde matrix $\boldsymbol{\mathcal{V}}(\boldsymbol{\xi}) \in \mathbb{C}^{N\times L_c}$, where $\boldsymbol{\xi} \triangleq [\xi_1, \xi_2, \ldots, \xi_{L_c}]^T \in \mathbb{C}^{L_c}$ is an arbitrary channel-zero vector, with $\xi_{q_1} \neq \xi_{q_2}$ for $q_1 \neq q_2 \in \{1, 2, \ldots, L_c\}$. To this aim, we have to preliminary characterize the set $\mathcal{B} \triangleq \{\boldsymbol{\psi} \in \mathbb{C}^N \,|\, \text{rank}[\boldsymbol{\Phi}(\boldsymbol{\xi},\boldsymbol{\psi})] = 2\,L_c \text{ for any } \boldsymbol{\xi} \in \mathbb{C}^{L_c}\}$, where $\boldsymbol{\Phi}(\boldsymbol{\xi},\boldsymbol{\psi}) \triangleq [\,\boldsymbol{\mathcal{V}}(\boldsymbol{\xi}), \boldsymbol{\Psi}(\boldsymbol{\psi})\,\boldsymbol{\mathcal{V}}^*(\boldsymbol{\xi})\,] \in \mathbb{C}^{N\times 2L_c}$. Observe that $\boldsymbol{\psi} \notin \mathcal{B}$ iff there exists a vector $\boldsymbol{\xi} \in \mathbb{C}^{L_c}$ such that $\text{rank}[\boldsymbol{\Phi}(\boldsymbol{\xi},\boldsymbol{\psi})] < 2\,L_c$, which, in its turns, holds iff there exists a *nonzero* vector $\boldsymbol{\alpha} \triangleq [\alpha_1, \alpha_2, \ldots, \alpha_{L_c}]^T \in \mathbb{C}^{L_c}$ such that $\boldsymbol{\mathcal{V}}(\boldsymbol{\xi})\,\boldsymbol{\alpha} - \boldsymbol{\Psi}(\boldsymbol{\psi})\,\boldsymbol{\mathcal{V}}^*(\boldsymbol{\xi})\,\boldsymbol{\alpha}^* = \mathbf{0}_N$. Based on this equivalence, we can state that, let $\boldsymbol{v}(\boldsymbol{\xi}) \triangleq \boldsymbol{\mathcal{V}}(\boldsymbol{\xi})\,\boldsymbol{\alpha} \in \mathbb{C}^N$ be a nonzero vector belonging to the range $\mathcal{R}[\boldsymbol{\mathcal{V}}(\boldsymbol{\xi})]$ of the Vandermonde matrix $\boldsymbol{\mathcal{V}}(\boldsymbol{\xi})$, vector $\boldsymbol{\psi} \notin \mathcal{B}$ iff there exists a vector $\boldsymbol{\xi} \in \mathbb{C}^{L_c}$ such that $\boldsymbol{\Psi}(\boldsymbol{\psi})\,\boldsymbol{v}^*(\boldsymbol{\xi}) = \boldsymbol{v}(\boldsymbol{\xi})$. Conversely, $\boldsymbol{\psi}$ belongs to $\mathcal{B}$ iff, for any $\boldsymbol{\xi} \in \mathbb{C}^{L_c}$, there is no vector $\boldsymbol{v}(\boldsymbol{\xi}) \in \mathcal{R}[\boldsymbol{\mathcal{V}}(\boldsymbol{\xi})] - \{\mathbf{0}_N\}$ such that $\boldsymbol{\Psi}(\boldsymbol{\psi})\,\boldsymbol{v}^*(\boldsymbol{\xi}) = \boldsymbol{v}(\boldsymbol{\xi})$. Owing to the structure of $\boldsymbol{\Psi}(\boldsymbol{\psi})$, one obtains the following Theorem:

Theorem 2: Let $\boldsymbol{\mathcal{V}}(\boldsymbol{\xi}) \in \mathbb{C}^{N\times L_c}$ be the Vandermonde matrix associated with $\boldsymbol{\xi} \triangleq [\xi_1, \xi_2, \ldots, \xi_{L_c}]^T \in \mathbb{C}^{L_c}$, where $\xi_{q_1} \neq \xi_{q_2}$ for $q_1 \neq q_2 \in \{1, 2, \ldots, L_c\}$. Condition A3 holds for any FIR channel of order L_c if, for any $\boldsymbol{\xi} \in \mathbb{C}^{L_c}$, there is no vector $\boldsymbol{v}(\boldsymbol{\xi}) \triangleq [v_0(\boldsymbol{\xi}), v_1(\boldsymbol{\xi}), \ldots, v_{N-1}(\boldsymbol{\xi})]^T \in \mathcal{R}[\boldsymbol{\mathcal{V}}(\boldsymbol{\xi})] - \{\mathbf{0}_N\}$ such that $\psi_m = 2\arg[v_m(\boldsymbol{\xi})] \pmod{2\pi}$, for any $m \in \{0, 1, \ldots, N-1\}$, with $\arg[v_m(\boldsymbol{\xi})]$ denoting the argument of $v_m(\boldsymbol{\xi})$.

Observe that the mth entry $v_m(\boldsymbol{\xi})$ of a vector $\boldsymbol{v}(\boldsymbol{\xi}) \in \mathcal{R}[\boldsymbol{\mathcal{V}}(\boldsymbol{\xi})]$ exhibits a particular structural property: $\boldsymbol{v}(\boldsymbol{\xi})$ belongs to $\mathcal{R}[\boldsymbol{\mathcal{V}}(\boldsymbol{\xi})]$ iff, for any $m \in \{0, 1, \ldots, N-1\}$, $v_m(\boldsymbol{\xi})$ can be expressed as a linear combination of the mth power of $\{\xi_q\}_{q=1}^{L_c}$ with coefficients $\{\alpha_q\}_{q=1}^{L_c}$ (independent of m), i.e., $v_m(\boldsymbol{\xi}) = \sum_{q=1}^{L_c} \alpha_q\,\xi_q^m$. Loosely speaking, Theorem 2 states that condition A3 is always satisfied if, for some $m \in \{0, 1, \ldots, N-1\}$, the precoding-dependent phase ψ_m *cannot* be expressed (mod 2π) as twice the argument of a complex number exhibiting the same structural property of $v_m(\boldsymbol{\xi})$. This suggests the following procedure for synthesizing an universal MIC-based precoding matrix: *Step 1*: select $N L_c$ distinct nonzero points $\{\chi_{m,q}\}_{q=1}^{L_c}$ on the complex plane such that $\chi_{0,q} \neq 1$ and, for $m \in \{1, 2, \ldots, N-1\}$, $\chi_{m,q} \neq \mu_q \, \chi_{m-1,q}$, with an arbitrary $\mu_q \in \mathbb{C}$; then, construct the complex numbers[4] $\chi_m = \sum_{q=1}^{L_c} \chi_{m,q}$, whose intrinsic structure is completely different from that of $v_m(\boldsymbol{\xi})$. *Step 2*: build the phases $\{\psi_m\}_{m=0}^{N-1}$ as $\psi_m = 2 \arg(\chi_m) \pmod{2\pi}$. *Step 3*: construct the universal precoding matrix $\boldsymbol{\mathcal{F}}_{\text{mic}} = \text{diag}[f(0), f(1), \ldots, f(N-1)]$, where $f(m) = \rho_m \, e^{j\,\theta_m}$, with $\theta_m = 0.5\,\psi_m + m\,\pi\,\delta$ and $\{\rho_m\}_{m=1}^{N}$ are arbitrarily chosen.

Remark 4. It is worthwhile to note that the proposed precoding algorithm *does not* require CSI at the transmitter, i.e., knowledge of the channel-zero locations. Moreover, as it is experimentally verified (see results of Fig. 1), our method is interestingly *not* affected by channel-order mismatches.

We present now the results of a Monte Carlo computer simulation. We considered an OQPSK system with block length $N = 16$, employing different precoding strategies at the transmitter. In particular, let $\widehat{L}_c$ denote an estimate of L_c, we synthesized three unitary diagonal matrices $\boldsymbol{\mathcal{F}}_{\text{univ}}$, $\widehat{\boldsymbol{\mathcal{F}}}_{\text{univ}}^{(+)}$ and $\widehat{\boldsymbol{\mathcal{F}}}_{\text{univ}}^{(-)}$ following steps 1–3, with $\widehat{L}_c = L_c$, $\widehat{L}_c = L_c + 1$ and $\widehat{L}_c = L_c - 1$, respectively. As a comparison, we considered a system without precoding and a system employing the diagonal precoding matrix proposed in [4, Eq. (41), $\rho = 0.5$]. At the receiver, we considered the following detection strategies: the WL-FIR-ZF block equalizer [see Theorem 1, with $\boldsymbol{\mathcal{Y}} = \mathbf{O}_{N \times 2(N-L_c)}$]; the IBI-free WL-FIR-MMSE block equalizer; the L-FIR-MMSE (i.e., $\boldsymbol{G}_2 = \mathbf{O}_{N \times N}$) block equalizer. The channel is a 3rd-order random FIR system, whose taps are modeled as i.i.d. complex proper zero-mean Gaussian random variables, with variance $\sigma_h^2 = 1/4$. The noise samples $w(n)$ are modeled as i.i.d. complex proper zero-mean Gaussian random variables, independent of $s(n)$, with variance σ_w^2. All the receivers are synthesized by assuming perfect knowledge of both the channel and the autocorrelation matrices required for the MMSE receiver synthesis. The results are obtained by carrying out 10^6 independent trials, each one using a different set of signal, noise, and channel realizations.

Fig. 1 depicts the average bit-error rate (BER) as a function of SNR $\triangleq [\sigma_s^2 \, \text{trace}(\boldsymbol{\mathcal{F}}^H \boldsymbol{\mathcal{F}})]/(N \, \sigma_w^2)$. First, as expected, the L-FIR-MMSE receiver performances are completely unsatisfactory, since they exhibit an unacceptable BER floor for high values of SNR, due to the lack of L-FIR-ZF solutions, whereas the performances of all WL receivers are superior and do not suffer from the previous limitation. Among the WL receivers, it can be observed that, when no precoding or precoding [4] is employed, the WL-FIR-MMSE equalizer significantly outperforms the WL-FIR-ZF one. Instead, the proposed universal precoding strategy, since it guarantees WL-FIR-ZF equalization for any FIR channel of order L_c, makes the WL-FIR-ZF receiver competitive with the WL-FIR-MMSE one, assuring also the best overall performances: this is advantageous since the WL-FIR-ZF equalizer is simpler to implement. Finally, the performances of the WL-FIR-ZF equalizers, with precoding matrices $\boldsymbol{\mathcal{F}}_{\text{univ}}$, $\widehat{\boldsymbol{\mathcal{F}}}_{\text{univ}}^{(+)}$ and $\widehat{\boldsymbol{\mathcal{F}}}_{\text{univ}}^{(-)}$, are comparable, corroborating thus the effectiveness of the proposed precoding procedure, even when the channel order is not exactly known[5].

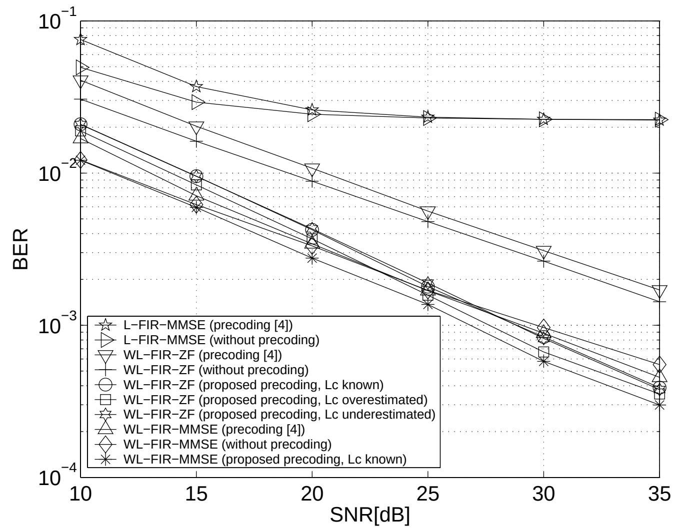

Fig. 1. BER versus SNR (OQPSK signaling, 10^6 channel realizations).

[4] In general, one can resort to any linear combination of $\{\chi_{m,q}\}_{q=1}^{L_c}$ with nonzero coefficients.

[5] Extensive simulation results for WL-FIR-ZF and WL-FIR-MMSE receivers, not reported here for brevity, have shown the robustness of the proposed precoding technique against inaccurate knowledge of L_c for a wide range of values of channel order mismatch.

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