FIR Zero-Forcing Multiuser Detection and Code Designs for Downlink MC-CDMA

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***Abstract*—This paper focuses on multiuser detection for multicarrier code-division multiple-access (MC-CDMA) systems, employing cyclic-prefixed (CP) or zero-padded (ZP) transmission techniques. For both systems, we consider either the classical linear finite-impulse response (L-FIR) receiving structures or, when the transmitted symbols are improper, the recently proposed widely linear FIR (WL-FIR) ones, which process both the received signal and its complex-conjugate version. With regard to both CP- and ZP-based downlink configurations, it is shown that, if the number of users does not exceed a given threshold and their codes are appropriately designed, L-FIR and WL-FIR universal zero-forcing (ZF) multiuser detectors can be synthesized, which, in the absence of noise, guarantee perfect symbol recovery for each user, regardless of the underlying frequency-selective channel. In particular, some spreading code examples are provided, which satisfy the design rules. Finally, numerical simulations are carried out to show that the theoretical considerations developed herein provide useful guidelines for practical MC-CDMA system designs.**

***Index Terms*—Linear and widely linear processing, multicarrier code-division multiple-access (MC-CDMA) systems, proper and improper random processes, spreading codes design, zero-forcing (ZF) multiuser detection.**

I. Introduction

MULTIUSER detection (MUD) for both direct-sequence (DS) [1] and multicarrier (MC) [2]–[4] code-division multiple-access (CDMA) systems has been extensively studied, since it allows one to achieve a dramatic performance improvement over simpler single-user detection schemes, in those environments where the multiple-access interference (MAI) is the predominant performance-limiting factor. This paper deals with MC-CDMA wireless networks employing *frequency-domain spreading* [3], which consists of copying each information symbol over the N subcarriers and multiplying it by a user-specific vector code. The choice of a multicarrier scheme is motivated by the fact that, at high data-rates (of the order of several hundred megabits per second), the common single-carrier DS-CDMA technology becomes impractical [5], due to both severe multipath-induced intersymbol interference (ISI) and synchronization difficulties. Indeed, MC-CDMA systems achieve ISI mitigation more efficiently, by transmitting with a lower data-rate over multiple subcarriers and introducing a suitable amount of redundancy in the transmitted data. Moreover, due to the lowered symbol rate, the synchronization task is easier in MC-CDMA networks, compared with a DS-CDMA system with similar processing gain. Furthermore, it has been shown in [6] that, at the expense of a reduced bandwidth efficiency, MUD techniques offer higher near-far resistance in MC-CDMA systems than in DS-CDMA ones. Finally, multiuser perfect symbol recovery (in the absence of noise) for DS-CDMA systems requires that certain technical conditions [7] on the channels and codes hold, which are difficult to check at the receiver.

Several MC-CDMA transmission schemes have been proposed in the literature. In conventional MC-CDMA systems employing frequency-domain spreading, after multiplying each information symbol by a user-specific vector code, the resulting vector is subject to inverse fast Fourier transform (IFFT) and, finally, a *cyclic prefix* (CP) of length L_{p} larger than the channel order L is inserted; at the receiver, the CP is discarded and the remaining part of the MC-CDMA symbol turns out to be free of interblock interference (IBI). An alternative precoding technique for achieving deterministic IBI cancellation, which has been originally proposed [8] for orthogonal frequency-division multiplexing (OFDM) systems, consists of replacing the CP with *zero padding* (ZP), by appending $L_{\mathrm{p}} > L$ zero symbols to each IFFT-precoded symbol block; in this case, IBI suppression is obtained without discarding any portion of the received signal. If the number of zero symbols is equal to the CP length, then CP- and ZP-based systems exhibit the same spectral efficiency. In the OFDM context, ZP precoding has been proposed since, unlike CP-based transmissions, it enables *linear* finite-impulse response (L-FIR) perfect symbol recovery, even when the channel transfer function has nulls on (or close to) some subcarriers. Compared with CP precoding, the price to pay for such a capability is the slightly increased receiver complexity and the larger power amplifier backoff. A recently proposed transmission scheme is adopted in the so-called generalized MC-CDMA (GMC-CDMA) systems [9], which subsume and extend previously proposed transmission techniques, such as [4], [10]–[12]. Contrary to CP- and ZP-MC-CDMA transmissions, wherein the spreading is performed over a single symbol (*symbol-spreading*), GMC-CDMA systems involve a significant modification of the conventional MC transmitter, whereby the information stream of each user is first collected in blocks and, then, each block is subject to both inner and outer spreading by means of two user-specific code matrices (*block-spreading*). At the expense of a reduced bandwidth efficiency, block spreading

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allows linear single-user equalization and MAI deterministic suppression, regardless of the multipath channel.

In this paper, we focus attention on linear and *widely linear* (WL) multiuser detection for both CP- and ZP-based downlink configurations (see Section II for the system models). We show that channel-irrespective FIR perfect symbol recovery is possible, i.e., *universal* FIR zero-forcing (ZF) designs can be carried out, without resorting to redundant block-spreading as in GMC-CDMA transmissions. To this aim, it should be observed that many L- and WL-MUD techniques, which were proposed in the DS-CDMA context, can be readily adapted to MC-CDMA systems. To suppress MAI with an affordable computational complexity and, simultaneously, achieve close-to-optimality performance (in the minimum bit-error-rate sense), one can resort to the FIR L-ZF (or linear decorrelating) and linear minimum mean-square error (L-MMSE) receivers [1]. More recently, it has been shown [13]–[18] that, by exploiting the possible *improper or noncircular* [19], [20] nature of the transmitted symbols, a property that is exhibited by many modulation schemes of practical interest [21]–[23], improved MAI suppression capabilities can be attained by adopting WL-FIR receiving structures [24], such as the WL-ZF and WL-MMSE receivers. It is worth noting that noncircularity of the transmitted symbols has also been exploited to improve single-carrier channel identification, equalization and synchronization [21]–[23], [25]–[28], as well as equalization and channel identification for single-user multicarrier systems [29] and blind source separation [30], [31].

With reference to FIR L-MUD receiving structures, it is known [4], [9] that perfect symbol recovery is guaranteed in a ZP-based downlink, for any FIR channel of order $L < L_{\rm p}$, as long as the number of users is smaller than the number of subcarriers (*underloaded systems*) and the code vectors are linearly independent. In general, a similar feature does not hold for CP-based transmissions. On the other hand, to the best of our knowledge, a detailed study of the conditions assuring FIR WL-MUD perfect symbol recovery in both CP- and ZP-based systems is lacking. Thus, the contribution of this paper is twofold. First, it is shown in Section III that, unlike CP-OFDM systems, universal L-ZF-MUD can be guaranteed even for the underloaded CP-MC-CDMA downlink, provided that the spreading codes are judiciously designed. Second, by gaining advantage of the results provided in Section III, it is further shown in Section IV that, if appropriate complex-valued spreading codes are employed, universal WL-ZF multiuser detectors can be designed even for *overloaded* CP-MC-CDMA and ZP-MC-CDMA systems.[1] Finally, Section V provides numerical results, aimed at corroborating the theoretical analyses carried out, whereas concluding remarks are drawn in Section VI.

A. Notations

Upper- and lower-case bold letters denote matrices and vectors; the superscripts $*, T, H, -1, -$ and $\dagger$ denote the conjugate, the transpose, the Hermitian (conjugate transpose), the inverse, the generalized (1)-inverse [33] and the Moore–Penrose generalized inverse [33] of a matrix; $\mathbb{C}, \mathbb{R}$ and $\mathbb{Z}$ are the fields of complex, real and integer numbers; $\mathbb{C}^n\,[\mathbb{R}^n]$ denotes the vector-space of all n-column vectors with complex [real] coordinates; similarly, $\mathbb{C}^{n\times m}$ $[\mathbb{R}^{n\times m}]$ denotes the vector-space of all the $n \times m$ matrices with complex [real] elements; the operators $\text{Re}\{\cdot\}$ and $\text{Im}\{\cdot\}$ stand for real and imaginary parts of any complex-valued matrix, vector or scalar; $\mathbf{0}_n, \mathbf{O}_{n\times m}$ and $\mathbf{I}_n$ denote the n-column zero vector, the $n \times m$ zero matrix and the $n \times n$ identity matrix; for any $\mathbf{a} \in \mathbb{C}^n, \|\mathbf{a}\|$ denotes the Euclidean norm; for any $\mathbf{A} \in \mathbb{C}^{n\times m}$, $\text{rank}(\mathbf{A}), N(\mathbf{A}), R(\mathbf{A})$ and $R^{\perp}(\mathbf{A})$ denote the rank of $\mathbf{A}$, the null space of $\mathbf{A}$, the column space of $\mathbf{A}$ and its orthogonal complement in $\mathbb{C}^n$; $\mathbf{A} = \text{diag}[\mathbf{A}_{11}, \mathbf{A}_{22}, \cdots, \mathbf{A}_{pp}] \in \mathbb{C}^{(np)\times(mp)}$, with $\mathbf{A}_{ii} \in \mathbb{C}^{n\times m}$, is a block diagonal matrix; the subscript c stands for continuous-time signals, $\text{E}[\cdot]$ denotes statistical averaging and, finally, $i \triangleq \sqrt{-1}$ denotes the imaginary unit.

II. Downlink Models For CP- And ZP-MC-CDMA Systems

Let us consider the downlink of a MC-CDMA system employing N subcarriers and accommodating J users. The information symbol $b_j(n)$ emitted by the jth user in the nth $(n \in \mathbb{Z})$ symbol interval multiplies the *frequency-domain* spreading code $\mathbf{c}_j \triangleq [c_j^{(0)}, c_j^{(1)}, \cdots, c_j^{(N-1)}]^T \in \mathbb{C}^N$, with $c_j^{(m)} \neq 0, \forall m \in \{0, 1, \ldots, N-1\}$ and $\forall j \in \{1, 2, \ldots, J\}$. The resulting N-length sequence is subject to the inverse discrete Fourier transform (IDFT), producing, thus, the block

$$\tilde{\mathbf{u}}_j(n) = \mathbf{W}_{\rm IDFT}\mathbf{c}_j b_j(n) \in \mathbb{C}^N \tag{1}$$

where $\mathbf{W}_{\rm IDFT} \in \mathbb{C}^{N\times N}$ denotes the IDFT matrix, with (m_1, m_2)th entry $1/(\sqrt{N})e^{i[(2\pi)/N](m_1-1)(m_2-1)}$, for $m_1, m_2 \in \{1, 2, \ldots, N\}$, and its inverse $\mathbf{W}_{\rm DFT} \triangleq \mathbf{W}_{\rm IDFT}^{-1} = \mathbf{W}_{\rm IDFT}^{H}$ defines the discrete Fourier transform (DFT) matrix. For such systems, two different linear precoding strategies [8] can be pursued: cyclic prefixing and zero padding.

Cyclic prefixing: This kind of precoding is adopted in conventional CP-MC-CDMA systems, wherein, after computing the IDFT, a CP of length $L_{\rm p} \ll N$ is inserted at the beginning of $\tilde{\mathbf{u}}_j(n)$, obtaining, thus, the vector

$$\mathbf{u}_{{\rm cp},j}(n) = \mathbf{T}_{\rm cp}\mathbf{W}_{\rm IDFT}\mathbf{c}_j b_j(n) \tag{2}$$

where $\mathbf{T}_{\rm cp} \triangleq [\mathbf{I}_{\rm cp}^T, \mathbf{I}_N]^T \in \mathbb{R}^{P\times N}$ and $P \triangleq L_{\rm p} + N$, with $\mathbf{I}_{\rm cp} \in \mathbb{R}^{L_{\rm p}\times N}$ built by drawing out the last $L_{\rm p}$ rows of the identity matrix $\mathbf{I}_N$.

Zero padding: This alternative precoding technique, which is employed in ZP-MC-CDMA networks, consists of extending $\tilde{\mathbf{u}}_j(n)$ by resorting to zero padding rather than CP insertion; specifically, after computing the IDFT, $L_{\rm p} \ll N$ trailing zeros are padded at the end of $\tilde{\mathbf{u}}_j(n)$, obtaining, thus, the vector

$$\mathbf{u}_{{\rm zp},j}(n) = \mathbf{T}_{\rm zp}\mathbf{W}_{\rm IDFT}\mathbf{c}_j b_j(n) \tag{3}$$

where $\mathbf{T}_{\rm zp} \triangleq [\mathbf{I}_N, \mathbf{O}_{N\times L_{\rm p}}]^T \in \mathbb{R}^{P\times N}$.

In either cases, the blocks $\mathbf{u}_{{\rm cp},j}(n)$ and $\mathbf{u}_{{\rm zp},j}(n)$ are subject to parallel-to-serial conversion, and the resulting sequences feed a digital-to-analog converter, operating at rate $1/T_c = P/T_s$,

[1] It is worthwhile to observe that overloaded systems are of practical interest [32], for example, in bandwidth-efficient multiuser communication, where the bandwidth is at a premium.

where T_s and T_c denote the symbol and the sampling period, respectively. In the downlink, all the users are synchronous and propagate through a common channel: throughout the paper, we assume that the baseband-equivalent frequency-selective channel is modeled as a linear time-invariant system, whose channel impulse response $g_c(t)$ (including transmitting filter, physical channel and receiving filter) is *complex-valued*, that is, neither $\text{Re}\{g_c(t)\}$ nor $\text{Im}\{g_c(t)\}$ vanish identically, and spans $L+1$ sampling periods, i.e., $g_c(t) \equiv 0, \forall t \notin [0, LT_c]$, where $g_c(0), g_c(LT_c) \neq 0$, with $L < P$ within one symbol interval. In this case, the discrete-time channel $g(\ell) \triangleq g_c(\ell T_c)$ turns out to be a FIR filter of order L, i.e., $g(\ell) \equiv 0, \forall \ell \notin \{0, 1, \ldots, L\}$, with $g(0), g(L) \neq 0$. Furthermore, we will assume that the channel order L is not exactly known, but is upper bounded by L_p, i.e., $L < L_\text{p}$.

In a CP-based system, the IBI is deterministically removed by discarding the first L_p samples of each P-dimensional received block. Indeed, after CP removal, the kth ($k \in \mathbb{Z}$) received symbol block $\mathbf{r}_\text{cp}(k) \in \mathbb{C}^N$ can be expressed (see, e.g., [4] and [8]) as

$$\mathbf{r}_\text{cp}(k) = \underbrace{\boldsymbol{\Theta}_\text{cp}\mathbf{W}_\text{IDFT}\mathbf{C}}_{\boldsymbol{\mathcal{G}}_\text{cp} \in \mathbb{C}^{N \times J}}\mathbf{b}(k) + \mathbf{v}_\text{cp}(k) = \boldsymbol{\mathcal{G}}_\text{cp}\mathbf{b}(k) + \mathbf{v}_\text{cp}(k) \tag{4}$$

where $\boldsymbol{\Theta}_\text{cp} \in \mathbb{C}^{N \times N}$ is the circulant [34] matrix having $\boldsymbol{\Omega}_\text{cp}\mathbf{g}$ as its first column, with $\boldsymbol{\Omega}_\text{cp} \triangleq [\mathbf{I}_{L_\text{p}}, \mathbf{O}_{L_\text{p} \times (N - L_\text{p})}]^T \in \mathbb{R}^{N \times L_\text{p}}$ and $\mathbf{g} \triangleq [g(0), g(1), \ldots, g(L), 0, \ldots, 0]^T \in \mathbb{C}^{L_\text{p}}$, the vector $\mathbf{b}(k) \triangleq [b_1(k), b_2(k), \ldots, b_J(k)]^T \in \mathbb{C}^J$ collects the symbols transmitted by the users, $\mathbf{C} \triangleq [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_J] \in \mathbb{C}^{N \times J}$ defines the frequency-domain *code matrix* and, finally, vector $\mathbf{v}_\text{cp}(k) \in \mathbb{C}^N$ accounts for thermal noise. In contrast, ZP-based precoding allows one to deterministically eliminate the IBI by retaining all the samples of each P-dimensional received block. Specifically, in a ZP-based system, the kth received symbol block $\mathbf{r}_\text{zp}(k) \in \mathbb{C}^P$ is given by (see, e.g., [4] and [8])

$$\mathbf{r}_\text{zp}(k) = \underbrace{\boldsymbol{\Theta}_\text{zp}\mathbf{W}_\text{IDFT}\mathbf{C}}_{\boldsymbol{\mathcal{G}}_\text{zp} \in \mathbb{C}^{P \times J}}\mathbf{b}(k) + \mathbf{v}_\text{zp}(k) = \boldsymbol{\mathcal{G}}_\text{zp}\mathbf{b}(k) + \mathbf{v}_\text{zp}(k) \tag{5}$$

where $\boldsymbol{\Theta}_\text{zp} \in \mathbb{C}^{P \times N}$ is the Toeplitz [34] matrix having $\boldsymbol{\Omega}_\text{zp}\mathbf{g}$ as first column, with $\boldsymbol{\Omega}_\text{zp} \triangleq [\mathbf{I}_{L_\text{p}}, \mathbf{O}_{L_\text{p} \times (P - L_\text{p})}]^T \in \mathbb{R}^{P \times L_\text{p}}$, and $[g(0), 0, \ldots, 0]$ as first row, whereas $\mathbf{v}_\text{zp}(k) \in \mathbb{C}^P$ accounts for thermal noise. For the sake of conciseness, we unify models (4) and (5) in the equivalent one

$$\mathbf{r}(k) = \boldsymbol{\mathcal{G}}\mathbf{b}(k) + \mathbf{v}(k), \quad \text{with} \quad \mathbf{r}(k), \mathbf{v}(k) \in \mathbb{C}^R \quad \text{and} \quad \boldsymbol{\mathcal{G}} \in \mathbb{C}^{R \times J} \tag{6}$$

where, for a CP-based system $\mathbf{r}(k) = \mathbf{r}_\text{cp}(k), \boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}_\text{cp}, \mathbf{v}(k) = \mathbf{v}_\text{cp}(k)$ with $R = N$, whereas, for a ZP-based system, $\mathbf{r}(k) = \mathbf{r}_\text{zp}(k), \boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}_\text{zp}, \mathbf{v}(k) = \mathbf{v}_\text{zp}(k)$, with $R = P$. Hereinafter, the following assumptions will be considered:

A1) the transmitted symbols $b_j(n)$ are modeled as mutually independent zero-mean and independent identically-distributed (i.i.d.) random sequences, with second-order moments $\sigma_b^2 \triangleq \text{E}[|b_j(n)|^2] > 0$ and $\varrho_b(n) \triangleq \text{E}[b_j^2(n)]$;

A2) the noise vector $\mathbf{v}(k)$ is a zero-mean wide-sense stationary complex *proper* [35], [36] white random process, which is independent of $b_j(n), \forall j \in \{1, 2, \ldots, J\}$, with autocorrelation matrix $\mathbf{R}_{\mathbf{vv}} \triangleq \text{E}[\mathbf{v}(k)\mathbf{v}^H(k)] = \sigma_v^2\mathbf{I}_R$.

As regards assumption A1), observe that, for some modulation formats [37], such as, e.g., M-PSK and M-QAM (with $M > 2$), the transmitted symbol sequences $\{b_j(n)\}_{j=1}^J$ are proper random processes [35], [36], i.e., $\varrho_b(n) = 0, \forall n \in \mathbb{Z}$. However, there exists a large family of modulation schemes of practical interest [21]–[23], such as, BPSK, DBPSK, M-ASK, OQPSK, OQAM, and binary CPM, MSK, GMSK, which turn out to be improper [19], [20], i.e., $\varrho_b(n) \neq 0$, for any $n \in \mathbb{Z}$.

III. Perfect Symbol Recovery For L-MUD

We address ZF detectability issues arising in FIR L-MUD [1], which can be used for both CP- and ZP-based systems, employing either proper or improper data symbols (although it is suboptimal in the latter case). These theoretical aspects strongly affect both the synthesis and the performance analysis of the L-ZF and L-MMSE multiuser detectors [1], [7].

Consider the problem of detecting the transmitted symbol $b_j(k)$ of the jth user from the received vector (6), with $j \in \{1, 2, \ldots, J\}$. To this purpose, a FIR L-MUD detector is defined by the input-output relationship $y_j(k) = \mathbf{f}_j^H\mathbf{r}(k)$, with $\mathbf{f}_j \in \mathbb{C}^R$, which is followed by a decision device. In the absence of noise, the perfect or ZF symbol recovery condition $y_j(k) = b_j(k)$ leads to the system of linear equations $\boldsymbol{\mathcal{G}}^H\mathbf{f}_j = \mathbf{e}_j$, where $\mathbf{e}_j \triangleq [\mathbf{0}_{j-1}^T, 1, \mathbf{0}_{J-j}^T]^T \in \mathbb{R}^J$, which is consistent (i.e., it admits at least one solution) for each user if and only if (iff) the *composite* channel matrix $\boldsymbol{\mathcal{G}}$ is full-column rank, i.e., $\text{rank}(\boldsymbol{\mathcal{G}}) = J$. Under this assumption, the *minimal norm* [33] solution of $\boldsymbol{\mathcal{G}}^H\mathbf{f}_j = \mathbf{e}_j$ is given by

$$\mathbf{f}_{\text{L-ZF},j} = \boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{G}}^H\boldsymbol{\mathcal{G}})^{-1}\mathbf{e}_j \tag{7}$$

which defines the L-ZF or linear decorrelating multiuser detector [1], [7]. In the presence of noise, the L-ZF receiver perfectly suppresses the MAI at the price of noise enhancement. To better counteract the noise, one can resort to the L-MMSE multiuser detector [1], [7], which is defined as

$$\mathbf{f}_{\text{L-MMSE},j} = \text{argmin}_{\mathbf{f}_j \in \mathbb{C}^R}\text{E}[|b_j(k) - y_j(k)|^2] = \sigma_b^2\mathbf{R}_{\mathbf{rr}}^{-1}\boldsymbol{\mathcal{G}}\mathbf{e}_j \tag{8}$$

where $\mathbf{R}_{\mathbf{rr}} \triangleq \text{E}[\mathbf{r}(k)\mathbf{r}^H(k)] \in \mathbb{C}^{R \times R}$ is the autocorrelation matrix of $\mathbf{r}(k)$ which, accounting for (6), and invoking assumptions A1) and A2), is given by

$$\mathbf{R}_{\mathbf{rr}} = \sigma_b^2\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{G}}^H + \sigma_v^2\mathbf{I}_R. \tag{9}$$

If $\boldsymbol{\mathcal{G}}$ is full-column rank, by resorting to the limit formula for the Moore–Penrose inverse [33], it can be seen that

$$\begin{aligned}\lim_{\sigma_v^2/\sigma_b^2 \to 0}\mathbf{f}_{\text{L-MMSE},j} &= \lim_{\sigma_v^2/\sigma_b^2 \to 0}\sigma_b^2(\sigma_b^2\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{G}}^H + \sigma_v^2\mathbf{I}_R)^{-1}\boldsymbol{\mathcal{G}}\mathbf{e}_j \\ &= \boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{G}}^H\boldsymbol{\mathcal{G}})^{-1}\mathbf{e}_j = \mathbf{f}_{\text{L-ZF},j}\end{aligned} \tag{10}$$

$\forall j \in \{1, 2, \ldots, J\}$, i.e., the L-MMSE receiver boils down to the L-ZF one.[2] In summary, the performance of the L-MMSE receiver in the high signal-to-noise (SNR) region strongly depends on the existence of L-ZF solutions: indeed, if $\boldsymbol{\mathcal{G}}$ is not full-column rank, the performance curve of the L-MMSE multiuser detector exhibits a marked bit-error-rate (BER) floor (see Section V), when $\sigma_v^2/\sigma_b^2 \to 0$. Motivated by this fact, the first step of our study consists of investigating whether the condition $\text{rank}(\boldsymbol{\mathcal{G}}) = J$ is satisfied, regardless of the frequency-selective channel.

As a matter of fact, for a ZP-based system [see (5)], the rank properties of $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}_{\text{zp}} = \boldsymbol{\Theta}_{\text{zp}} \mathbf{W}_{\text{IDFT}} \mathbf{C}$ are easily characterized, since the Toeplitz matrix $\boldsymbol{\Theta}_{\text{zp}}$ is full-column rank for any FIR channel of order L [4], [9], [10]. Indeed, owing to nonsingularity of $\mathbf{W}_{\text{IDFT}}$, it results that $\text{rank}(\boldsymbol{\mathcal{G}}_{\text{zp}}) = \text{rank}(\mathbf{C})$: as stated in [9], the composite channel matrix $\boldsymbol{\mathcal{G}}_{\text{zp}}$ is always full-column rank and, thus, channel-irrespective L-FIR perfect symbol recovery is possible iff the vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_J$ are linearly independent, that is, $\mathbf{C}$ is full-column rank. To this aim, one can for example use Walsh–Hadamard (WH) spreading codes, which are widely used in CDMA systems. It is worth noting that condition $\text{rank}(\mathbf{C}) = J$ imposes that the number of users be smaller than or equal to the number of subcarriers, i.e., $J \leq N$: strictly speaking, L-ZF-MUD is exclusively targeted at underloaded systems. On the other hand, for a CP-based system [see (4)], the linear independence of the code vectors is not sufficient to assure that $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}_{\text{cp}} = \boldsymbol{\Theta}_{\text{cp}} \mathbf{W}_{\text{IDFT}} \mathbf{C}$ be always full-column rank since, unlike $\boldsymbol{\Theta}_{\text{zp}}$, the circulant matrix $\boldsymbol{\Theta}_{\text{cp}}$ turns out to be singular for some FIR channels. However, after characterizing the rank properties of $\boldsymbol{\mathcal{G}}_{\text{cp}}$, we will show in Section III-A that, through appropriate design of user codes, the condition $\text{rank}(\boldsymbol{\mathcal{G}}_{\text{cp}}) = J$ can be guaranteed regardless of the underlying frequency-selective channel. Hence, channel-irrespective L-ZF-MUD is possible not only in a ZP-based system, but also in a CP-based one.

A. Rank Characterization of $\boldsymbol{\mathcal{G}}_{\text{cp}}$ and Universal Code Design for L-ZF-MUD

With reference to a CP-based system, let us study the rank properties of $\boldsymbol{\mathcal{G}}_{\text{cp}} = \boldsymbol{\Theta}_{\text{cp}} \mathbf{W}_{\text{IDFT}} \mathbf{C}$, whose characterization is more cumbersome than that of $\boldsymbol{\mathcal{G}}_{\text{zp}}$. Preliminarily, observe that $\boldsymbol{\mathcal{G}}_{\text{cp}}$ is full-column rank only if the number J of users is not larger than the number of subcarriers N, i.e., $J \leq N$. Thus, as for a ZP-based system, L-ZF-MUD is confined only to underloaded CP-based systems. Furthermore, by resorting to standard eigenstructure concepts [4], [34], one has $\boldsymbol{\Theta}_{\text{cp}} = \mathbf{W}_{\text{IDFT}} \boldsymbol{\Gamma}_{\text{cp}} \mathbf{W}_{\text{DFT}}$, where the diagonal entries of $\boldsymbol{\Gamma}_{\text{cp}} \triangleq \text{diag}[\gamma_{\text{cp}}(0), \gamma_{\text{cp}}(1), \ldots, \gamma_{\text{cp}}(N-1)] \in \mathbb{C}^{N \times N}$ are the values of the channel transfer function $G(z) \triangleq \Sigma_{\ell=0}^{L} g(\ell) z^{-\ell}$ evaluated at the subcarriers $z_m \triangleq e^{i(2\pi/N)m}$, i.e., $\gamma_{\text{cp}}(m) = G(z_m), \forall m \in \{0, 1, \ldots, N-1\}$. Henceforth, one obtains that $\boldsymbol{\mathcal{G}}_{\text{cp}} = \mathbf{W}_{\text{IDFT}} \boldsymbol{\Gamma}_{\text{cp}} \mathbf{C}$ and, since $\mathbf{W}_{\text{IDFT}}$ is nonsingular, it follows that $\text{rank}(\boldsymbol{\mathcal{G}}_{\text{cp}}) = \text{rank}(\boldsymbol{\Gamma}_{\text{cp}} \mathbf{C})$. The full-column rank property of matrix $\boldsymbol{\mathcal{G}}_{\text{cp}}$ is characterized by the following theorem.

Theorem 1 (Rank characterization of $\boldsymbol{\mathcal{G}}_{\text{cp}}$): If $\mathbf{C}$ is full-column rank and the channel transfer function $G(z)$ has $0 \leq M_z \leq L$ distinct zeros at $z_{m_1} = e^{i(2\pi/N)m_1}$, $z_{m_2} = e^{i(2\pi/N)m_2}, \ldots, z_{m_{M_z}} = e^{i(2\pi/N)m_{M_z}}$, with $m_1 \neq m_2 \neq \cdots \neq m_{M_z} \in \{0, 1, \ldots, N-1\}$, then the composite channel matrix $\boldsymbol{\mathcal{G}}_{\text{cp}}$ is full-column rank iff $[\mathbf{C}, \mathbf{S}_z] \in \mathbb{C}^{N \times (J+M_z)}$ is full-column rank, where $\mathbf{S}_z \triangleq [\mathbf{1}_{m_1}, \mathbf{1}_{m_2}, \ldots, \mathbf{1}_{m_{M_z}}] \in \mathbb{R}^{N \times M_z}$ is a full-column rank matrix, with $\mathbf{1}_m$ denoting the $(m+1)$th column of $\mathbf{I}_N$.

Proof: Let us consider the case when the channel transfer function $G(z)$ has $0 \leq M_z \leq L$ distinct zeros on the subcarriers $z_{m_1} = e^{i(2\pi/N)m_1}, z_{m_2} = e^{i(2\pi/N)m_2}, \ldots, z_{m_{M_z}} = e^{i(2\pi/N)m_{M_z}}$, with $m_1 \neq m_2 \neq \cdots \neq m_{M_z} \in \{0, 1, \ldots, N-1\}$. In this case, one has $\gamma_{\text{cp}}(m_1) = \gamma_{\text{cp}}(m_2) = \cdots = \gamma_{\text{cp}}(m_{M_z}) = 0$ and, thus, the diagonal matrix $\boldsymbol{\Gamma}_{\text{cp}}$ is singular with $\text{rank}(\boldsymbol{\Gamma}_{\text{cp}}) = N - M_z$. In its turn, this implies that $\boldsymbol{\mathcal{G}}_{\text{cp}}$ may be rank deficient even if the code vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_J$ are linearly independent, i.e., $\mathbf{C}$ is full-column rank. Indeed, under the assumptions that $J \leq N$ and $\text{rank}(\mathbf{C}) = J$, the matrix $\boldsymbol{\Gamma}_{\text{cp}} \mathbf{C}$ is full-column rank iff [33] $\mathcal{N}(\boldsymbol{\Gamma}_{\text{cp}}) \cap \mathcal{R}(\mathbf{C}) = \mathbf{0}_N$. The null space of $\boldsymbol{\Gamma}_{\text{cp}}$ can be readily characterized: an arbitrary vector $\boldsymbol{\mu} \in \mathbb{C}^N$ belongs to $\mathcal{N}(\boldsymbol{\Gamma}_{\text{cp}})$ iff there exists a vector $\boldsymbol{\beta} \in \mathbb{C}^{M_z}$ such that $\boldsymbol{\mu} = \mathbf{S}_z \boldsymbol{\beta}$. Hence, an arbitrary vector $\boldsymbol{\mu} \in \mathcal{N}(\boldsymbol{\Gamma}_{\text{cp}})$ also belongs to the subspace $\mathcal{R}(\mathbf{C})$ iff there exists a vector $\boldsymbol{\alpha} \in \mathbb{C}^J$ such that $\mathbf{S}_z \boldsymbol{\beta} = \mathbf{C} \boldsymbol{\alpha}$. As a consequence, condition $\mathcal{N}(\boldsymbol{\Gamma}_{\text{cp}}) \cap \mathcal{R}(\mathbf{C}) = \mathbf{0}_N$ holds iff the system of equations $\mathbf{C}\boldsymbol{\alpha} - \mathbf{S}_z \boldsymbol{\beta} = \mathbf{0}_N$ admits the unique solution $\boldsymbol{\alpha} = \mathbf{0}_J$ and $\boldsymbol{\beta} = \mathbf{0}_{M_z}$. It can be seen [34] that this happens iff the matrix $[\mathbf{C}, \mathbf{S}_z] \in \mathbb{C}^{N \times (J+M_z)}$ turns out to be full-column rank. ■

Some remarks are now in order concerning immediate implications of Theorem 1.

Remark 1: As aforementioned in the proof of Theorem 1, $\boldsymbol{\mathcal{G}}_{\text{cp}}$ may be rank deficient even if $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_J$ are linearly independent, i.e., $\mathbf{C}$ is full-column rank. However, if $G(z)$ has no zeros (i.e., $M_z = 0$) on the subcarriers $\{z_m\}_{m=0}^{N-1}$, that is, $\gamma_{\text{cp}}(m) \neq 0, \forall m \in \{0, 1, \ldots, N-1\}$, it results that $\boldsymbol{\Gamma}_{\text{cp}}$ is nonsingular and, consequently, $\text{rank}(\boldsymbol{\mathcal{G}}_{\text{cp}}) = \text{rank}(\mathbf{C})$. In other words, for a CP-based system, only if $G(z)$ has no zeros on the used subcarriers, the linear independence of the vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_J$ becomes a necessary *and* sufficient condition for assuring the full-column rank property of $\boldsymbol{\mathcal{G}}_{\text{cp}}$. In this case, both CP-based and ZP-based systems are able to support up to N active users.

Remark 2: Unlike conventional CP-OFDM systems [4], the presence of channel zeros on some subcarriers does not prevent perfect symbol recovery. This result stems from the fact that, in MC-CDMA systems with frequency-domain spreading, each symbol is transmitted in *parallel* on all the subcarriers; therefore, if the $\overline{m}$th subcarrier is hit by a channel zero, i.e., $\gamma_{\text{cp}}(\overline{m}) = 0$, the transmitted symbol $b_j(k)$ can still be recovered from the other subcarriers. In contrast, in CP-OFDM systems, wherein each subcarrier conveys a different data symbol, if $G(z)$ exhibits a zero on a used subcarrier, the symbol transmitted on that subcarrier is permanently lost [4], [10].

[2] More generally, when $\boldsymbol{\mathcal{G}}$ is possibly rank-deficient, it results that $\lim_{\sigma_v^2/\sigma_b^2 \to 0} \mathbf{f}_{\text{L-MMSE},j} = (\boldsymbol{\mathcal{G}}^H)^\dagger \mathbf{e}_j \triangleq \mathbf{f}_{\text{L-LS},j}$, i.e., the L-MMSE detector ends up to the minimal-norm least-squares solution [33] of $\boldsymbol{\mathcal{G}}^H \mathbf{f}_j = \mathbf{e}_j$ [note that, when $\boldsymbol{\mathcal{G}}$ is full-column rank, one has $\mathbf{f}_{\text{L-LS},j} = \mathbf{f}_{\text{L-ZF},j}$ from (7)].

Remark 3: The presence of channel zeros located at some subcarriers cuts down on the capacity[3] of a CP-based system. In fact, since the full-column rank property of $\boldsymbol{\mathcal{G}}_{\rm cp}$ is a necessary and sufficient condition for perfect recovery of $\mathbf{b}(k)$ in the absence of noise, Theorem 1 allows one to determine the maximum number of users that can be supported by a CP-based system employing L-ZF-MUD. More precisely, condition $\text{rank}(\boldsymbol{\mathcal{G}}_{\rm cp}) = J$ amounts to $\text{rank}([\mathbf{C}, \mathbf{S}_z]) = J + M_z$, which necessarily requires that $J \leq N - M_z$, with $0 < M_z \leq L < L_{\rm p} \ll N$. It follows that the number of active users that can be supported through L-ZF-MUD is decremented by one unit for any additional zero on the subcarriers[4]. In this case, the capacity of a CP-based downlink is smaller than that of a ZP-based system, which is equal to N regardless of the channel-zero configuration. In the worst case when $M_z = L$, i.e., all the channel zeros are located at the subcarriers, the maximum number of allowable users in a CP-based downlink is equal to $N - L$.

Theorem 1 evidences that, in contrast with ZP-based systems, the full-column rank property of $\boldsymbol{\mathcal{G}}_{\rm cp}$ depends not only on the linear independence of $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_J$, but also on the presence of channel zeros located at the subcarriers $\{z_m\}_{m=0}^{N-1}$, whose number M_z and locations $m_1, m_2, \ldots, m_{M_z}$ are *unknown* at the receiver. In other words, by imposing the unique constraint that $\mathbf{C}$ be full-column rank, perfect symbol recovery in a CP-based system explicitly depends on the channel impulse response. However, the usefulness of Theorem 1 goes beyond this aspect and, most importantly, it allows us to provide universal code designs, assuring that $\boldsymbol{\mathcal{G}}_{\rm cp}$ is full-column rank for *any* possible configuration of the channel zeros. To this aim, on the basis of Theorem 1, observing that $0 \leq M_z \leq L$ and any subset of linearly independent vectors is constituted by linearly independent vectors, we can state the following universal design constraint for the user codes in a CP-based system.

Condition $\mathrm{D}_{\rm cp}$ *(Universal Code Design for L-ZF-MUD in CP-MC-CDMA):* Under the assumption that $\mathbf{C}$ is full-column rank, no linear combination of the columns of $\mathbf{C}$ can be expressed as linear combinations of the L distinct vectors $\mathbf{1}_{m_1}, \mathbf{1}_{m_2}, \ldots, \mathbf{1}_{m_L}$, for *any* $\{m_1, m_2, \ldots, m_L\} \subset \{0, 1, \ldots, N-1\}$. Equivalently, require that

$$\text{rank}([\mathbf{C}, \mathbf{S}_{\rm univ}]) = J + L \qquad \forall \{m_1, m_2, \ldots, m_L\} \subset \{0, 1, \ldots, N-1\} \quad (11)$$

where $\mathbf{S}_{\rm univ} \triangleq [\mathbf{1}_{m_1}, \mathbf{1}_{m_2}, \ldots, \mathbf{1}_{m_L}] \in \mathbb{R}^{N \times L}$ is a full-column rank matrix.

By virtue of Theorem 1, the composite channel matrix $\boldsymbol{\mathcal{G}}_{\rm cp}$ turns out to be full-column rank for *any* FIR channel of order $L < L_{\rm p}$ iff the code design constraint $\mathrm{D}_{\rm cp}$ is fulfilled. Observe that $\mathrm{D}_{\rm cp}$ is stronger than condition $\text{rank}(\mathbf{C}) = J$. In fact, $\mathrm{D}_{\rm cp}$ implies that $\text{rank}(\mathbf{C}) = J$, whereas $\text{rank}(\mathbf{C}) = J$ does not imply $\mathrm{D}_{\rm cp}$. It is also apparent from (11) that, since $[\mathbf{C}, \mathbf{S}_{\rm univ}] \in \mathbb{C}^{N \times (J+L)}$, fulfillment of $\mathrm{D}_{\rm cp}$ imposes that no more than $N - L$ users can be handled by a CP-based system. Furthermore, it is worth noting that common WH spreading codes do not satisfy (11). To show this, as a simple counterexample, consider the case of two users (i.e., $J = 2$), which employs the following 8-length WH codes $\mathbf{c}_1 = [1, -1, 1, -1, 1, -1, 1, -1]^T$ and $\mathbf{c}_2 = [1, 1, -1, -1, 1, 1, -1, -1]^T$, obtained by picking the second and third columns of the Hadamard matrix of order $N = 8$. In this case, it is easily verified that $\mathbf{c}_1 + \mathbf{c}_2 = [2, 0, 0, -2, 2, 0, 0, -2]^T$. Hence, if the channel transfer function $G(z)$ has $M_z = 4$ zeros on the subcarriers z_0, z_3, z_4 and z_7, the corresponding matrix $\boldsymbol{\mathcal{G}}_{\rm cp}$ is not full-column rank, since a particular linear combination of $\mathbf{c}_1$ and $\mathbf{c}_2$ (i.e., the vector $\mathbf{c}_1 + \mathbf{c}_2$) can be expressed as the following linear combination $2\mathbf{1}_0 - 2\mathbf{1}_3 + 2\mathbf{1}_4 - 2\mathbf{1}_7$ of the vectors $\mathbf{1}_0, \mathbf{1}_3, \mathbf{1}_4$ and $\mathbf{1}_7$. Hence, WH spreading codes do not guarantee $\boldsymbol{\mathcal{G}}_{\rm cp}$ to be full-column rank for any FIR channel of order $L < L_{\rm p}$.

To design codes that instead fulfill $\mathrm{D}_{\rm cp}$, it is convenient to give an alternative interpretation of (11). Since it results [38] that $\text{rank}([\mathbf{C}, \mathbf{S}_{\rm univ}]) = \text{rank}(\mathbf{S}_{\rm univ}) + \text{rank}[(\mathbf{I}_N - \mathbf{S}_{\rm univ}\mathbf{S}_{\rm univ}^{-})\mathbf{C}]$, with $\text{rank}(\mathbf{S}_{\rm univ}) = L$ and $\mathbf{S}_{\rm univ}^{-} = \mathbf{S}_{\rm univ}^{T}$ [33], it follows that $\text{rank}([\mathbf{C}, \mathbf{S}_{\rm univ}]) = J + L$ holds iff $\text{rank}[(\mathbf{I}_N - \mathbf{S}_{\rm univ}\mathbf{S}_{\rm univ}^{T})\mathbf{C}] = J$. It can be verified by direct inspection that all the L rows of the matrix $(\mathbf{I}_N - \mathbf{S}_{\rm univ}\mathbf{S}_{\rm univ}^{T})\mathbf{C}$ located in the positions $m_1 + 1, m_2 + 1, \ldots, m_L + 1$ are zero (all the entries are equal to zero), whereas the $N - L$ remaining ones coincide with the corresponding rows of $\mathbf{C}$. Consequently, the condition $\text{rank}[(\mathbf{I}_N - \mathbf{S}_{\rm univ}\mathbf{S}_{\rm univ}^{T})\mathbf{C}] = J$, for any $\{m_1, m_2, \ldots, m_L\} \subset \{0, 1, \ldots, N-1\}$, is equivalent to state that, among any $N - L$ rows of $\mathbf{C}$, a set of $J \leq N - L$ linearly independent rows can be selected. More formally, $\mathrm{D}_{\rm cp}$ can be equivalently restated as follows.

Condition $\mathrm{D}_{\rm cp}$ *(Reformulation):* Let vector $\boldsymbol{\omega}_\ell^T \triangleq [c_1^{(\ell)}, c_2^{(\ell)}, \ldots, c_J^{(\ell)}] \in \mathbb{C}^{1 \times J}$ denote the $(\ell + 1)$th row of $\mathbf{C}$, with $\ell \in \{0, 1, \ldots, N-1\}$; for any $\{m_1, m_2, \ldots, m_L\} \subset \{0, 1, \ldots, N-1\}$, there exists a subset of $J \leq N - L$ distinct indices $\{\ell_1, \ell_2, \ldots, \ell_J\} \subset \{0, 1, \ldots, N-1\} - \{m_1, m_2, \ldots, m_L\}$ such that the vectors $\boldsymbol{\omega}_{\ell_1}, \boldsymbol{\omega}_{\ell_2}, \ldots, \boldsymbol{\omega}_{\ell_J}$ are linearly independent.

It is worthwhile to observe that condition $\mathrm{D}_{\rm cp}$ does not uniquely specify $\mathbf{C}$ and, thus, different universal codes can be built. For instance, condition $\mathrm{D}_{\rm cp}$ can be accomplished by imposing that each row of $\mathbf{C}$ be a Vandermonde-like vector. Specifically, let us select $N \geq J + L$ nonzero numbers $\{\rho_\ell\}_{\ell=0}^{N-1}$ and build the code vectors $\mathbf{c}_j$ as

$$\mathbf{c}_j = \frac{1}{\sqrt{\chi_j}} \left[\rho_0^j, \rho_1^j, \ldots, \rho_{N-1}^j\right]^T, \quad \forall j \in \{1, 2, \ldots, J\} \quad (12)$$

where the normalization by $1/\sqrt{\chi_j}$ has been introduced to ensure that $\|\mathbf{c}_j\|^2 = 1$ for each user. Relying on the properties of Vandermonde vectors [34], it can be easily verified that,

[3]Hereinafter, with the term "capacity" (not to be interpreted in the information-theoretic sense), we refer to the maximum number of users for which perfect symbol recovery is possible in the absence of noise.

[4]It is worth noting that, unlike L-ZF universal multiuser detectors, which do not exist for $J > N - M_z$, the L-MMSE multiuser detector can still be synthesized in the presence of noise even when $J > N - M_z$. However, as previously remarked, its performance is unsatisfactory in this case (see also Section V). Thus, $N - M_z$ also represents the maximum number of users that a CP-based system can reliably manage when L-MMSE-MUD is employed at the receiver.

provided that $\rho_0 \neq \rho_1 \neq \cdots \neq \rho_{N-1}$, any J rows of $\mathbf{C}$ are linearly independent, thus satisfying D_{cp}. An advantage of choosing the spreading vectors as in (12) is that, in this way, the code matrix $\mathbf{C}$ is uniquely characterized only by the N parameters $\{\rho_\ell\}_{\ell=0}^{N-1}$. For example, such numbers can be chosen equispaced on the unit circle, by setting $\rho_\ell = e^{-i(2\pi)/(N)\ell}, \forall \ell \in \{0, 1, \ldots, N-1\}$, thus, obtaining

$$\mathbf{c}_j = \frac{1}{\sqrt{N}} \left[1, e^{-i\frac{2\pi}{N}j}, \ldots, e^{-i\frac{2\pi}{N}(N-1)j}\right]^T \tag{13}$$

$\forall j \in \{1, 2, \ldots, J\}$. In this case, the spreading vector $\mathbf{c}_j$ turns out to be a Vandermonde (VM) vector (up to the power-controlling constant $1/\sqrt{N}$) and the columns of the resulting code matrix $\mathbf{C}$ coincide with some columns of the N-point DFT matrix $\mathbf{W}_{\mathrm{DFT}}$. Obviously, since the VM code vectors (13) are linearly independent by construction, they also guarantee the existence of L-ZF solutions for any FIR channel of order $L < L_{\mathrm{p}}$ in underloaded ZP-based systems. It should be observed that VM spreading has also been employed in GMC-CDMA systems [9], [12], wherein, however, a different code matrix (instead of a code vector) is assigned to each user in order to perform block-symbol spreading (instead of symbol-spreading). As a side comment, we conclude this section with the following remark.

Remark 4: Since the channel order L is seldom known in practice, one must resort to the upper bound $L < L_{\mathrm{p}}$ for synthesizing $\mathbf{C}$, i.e., one should use L_{p} instead of L in condition D_{cp}. So doing, the allowable number of users must obey $J \leq N - L_{\mathrm{p}}$, which is a more restrictive limit than $J \leq N - L$. In other words, requiring that the composite channel matrix $\boldsymbol{\mathcal{G}}_{\mathrm{cp}}$ be full-column rank for *any* FIR channel of order $L < L_{\mathrm{p}}$ poses a stronger limitation on system capacity.

IV. Perfect Symbol Recovery for WL-MUD

With reference to the unified model (6), when the information symbols are improper, L-MUD does not fully exploit the second-order statistics (SOS) of the received vector $\mathbf{r}(k)$. Indeed, it does not take into account the *conjugate* autocorrelation matrix [19] $\mathbf{R}_{\mathbf{rr}^*}(k) \triangleq \mathrm{E}[\mathbf{r}(k)\mathbf{r}^T(k)] \in \mathbb{C}^{R \times R}$ of $\mathbf{r}(k)$ which, invoking assumptions A1) and A2), can be written as $\mathbf{R}_{\mathbf{rr}^*}(k) = \varrho_b(k)\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{G}}^T$. In fact, for the problem at hand, $\mathbf{r}(k)$ turns out to be an improper random vector iff the sequences $\{b_j(k)\}_{j=1}^J$ are improper random processes [35], i.e., $\varrho_b(k) \neq 0$, for any $k \in \mathbb{Z}$. Observe that this condition does not hold for modulation formats, such as M-PSK and M-QAM (with $M > 2$). In these cases, it results that $\varrho_b(k) = 0$, for any $k \in \mathbb{Z}$, and, therefore, L-MUD fully exploits the SOS of $\mathbf{r}(k)$ for such modulation formats. However, as aforementioned, the symbols $b_j(k)$ are improper in a large number of digital modulation schemes [22], [23], including all the real-valued symbol sequences $b_j(k)$, such as BPSK, DBPSK, M-ASK, and many conjugate symmetric complex-valued symbol constellations, such as OQPSK, OQAM, and binary CPM, MSK, GMSK. In all these cases, the improper nature of $b_j(k)$ can be seen as the consequence of a linear deterministic dependence existing between $b_j(k)$ and its conjugate version $b_j^*(k)$, i.e., $b_j^*(k) = e^{i2\pi\xi k} b_j(k)$, for any $k \in \mathbb{Z}$ and for *any* realization of $b_j(k)$. Real modulation schemes fulfill the previous relation with $\xi = 0$, i.e., $b_j^*(k) = b_j(k)$, which implies that $\varrho_b(k) = \sigma_b^2$, for any $k \in \mathbb{Z}$, whereas for complex modulation formats, such as OQPSK, OQAM, and MSK, this relation is satisfied [22], [23] with $\xi = 1/2$, i.e., $b_j^*(k) = (-1)^k b_j(k)$, which implies that $\varrho_b(k) = \sigma_b^2(-1)^k$, for any $k \in \mathbb{Z}$. In the latter case, the process $b_j(k)$ turns out to be also (wide-sense) second-order conjugate cyclostationary [39] with period 2.

To conveniently exploit the improper nature of the transmitted symbols, WL-MUD structures, which are characterized [24] by the input-output relationship

$$w_j(k) = \mathbf{f}_{j,1}^H \mathbf{r}(k) + \mathbf{f}_{j,2}^H \mathbf{r}^*(k) \tag{14}$$

with $\mathbf{f}_{j,1}, \mathbf{f}_{j,2} \in \mathbb{C}^R$, have been proposed in [14]–[18], under the assumption that the users adopt BPSK modulation. It has been proven both theoretically [16] and experimentally that WL-MUD can achieve a significant performance gain and capacity improvement over L-MUD. However, as previously mentioned, the WL approach is not confined to real-valued modulations and, thus, a slight generalization of the classical WL-FIR estimator [24] has to be considered. To this aim, observe that the standard WL-FIR input-output relationship can be rewritten as $w_j(k) = \overline{\mathbf{f}}_j^H \tilde{\mathbf{z}}(k)$, where $\overline{\mathbf{f}}_j \triangleq [\mathbf{f}_{j,1}^T, \mathbf{f}_{j,2}^T]^T \in \mathbb{C}^{2R}$ and the *augmented* vector $\tilde{\mathbf{z}}(k) \triangleq [\mathbf{r}^T(k), \mathbf{r}^H(k)]^T \in \mathbb{C}^{2R}$ is obtained by stacking $\mathbf{r}(k)$ and its complex conjugate version $\mathbf{r}^*(k)$. Moreover, note that, with reference to the aforementioned improper modulations techniques, the following linear deterministic relationship holds: $\mathbf{b}^*(k) = e^{i2\pi\xi k}\mathbf{b}(k)$, for any $k \in \mathbb{Z}$ which, substituted in (6), yields $\mathbf{r}^*(k) = e^{i2\pi\xi k}\boldsymbol{\mathcal{G}}^*\mathbf{b}(k) + \mathbf{v}^*(k)$. The latter relation shows that the (possible) conjugate cyclostationarity of $b_j(k)$ can be deterministically compensated for by performing a *derotation* [21] of $\mathbf{r}^*(k)$ before evaluating $\tilde{\mathbf{z}}(k)$, that is, by considering the modified input-output relationship

$$w_j(k) = \mathbf{f}_{j,1}^H \mathbf{r}(k) + \mathbf{f}_{j,2}^H \mathbf{r}^*(k) e^{-i2\pi\xi k} = \overline{\mathbf{f}}_j^H \mathbf{z}(k) \tag{15}$$

where the augmented and derotated vector $\mathbf{z}(k) \in \mathbb{C}^{2R}$ is given by

$$\mathbf{z}(k) \triangleq \begin{bmatrix} \mathbf{r}(k) \\ \mathbf{r}^*(k)e^{-i2\pi\xi k} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\mathcal{G}} \\ \boldsymbol{\mathcal{G}}^* \end{bmatrix}}_{\boldsymbol{\mathcal{H}} \in \mathbb{C}^{2R \times J}} \mathbf{b}(k) + \underbrace{\begin{bmatrix} \mathbf{v}(k) \\ \mathbf{v}^*(k)e^{-i2\pi\xi k} \end{bmatrix}}_{\mathbf{w}(k) \in \mathbb{C}^{2R}} = \boldsymbol{\mathcal{H}}\mathbf{b}(k) + \mathbf{w}(k). \tag{16}$$

In the absence of noise, the ZF condition $w_j(k) = b_j(k)$ leads to the system of linear equations $\boldsymbol{\mathcal{H}}^H \overline{\mathbf{f}}_j = \mathbf{e}_j$, which is consistent for each user iff the augmented channel matrix $\boldsymbol{\mathcal{H}}$ is full-column rank, i.e., $\mathrm{rank}(\boldsymbol{\mathcal{H}}) = J$; under this assumption, the *minimal norm* [33] solution of $\boldsymbol{\mathcal{H}}^H \overline{\mathbf{f}}_j = \mathbf{e}_j$ is

$$\overline{\mathbf{f}}_{\mathrm{WL-ZF},j} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{H}}^H\boldsymbol{\mathcal{H}})^{-1}\mathbf{e}_j \tag{17}$$

which defines the WL-ZF or WL decorrelating multiuser detector. In the presence of noise, one can more suitably resort to the WL-MMSE multiuser detector [14]–[16], which is defined as

$$\begin{aligned}\bar{\mathbf{f}}_{\mathrm{WL-MMSE},j} &= \arg\min_{\bar{\mathbf{f}}_j \in \mathbb{C}^R} \mathrm{E}[|b_j(k) - w_j(k)|^2] \\ &= \sigma_b^2 \mathbf{R}_{\mathbf{zz}}^{-1} \boldsymbol{\mathcal{H}} \mathbf{e}_j \end{aligned} \tag{18}$$

where $\mathbf{R}_{\mathbf{zz}} \triangleq \mathrm{E}[\mathbf{z}(k)\mathbf{z}^H(k)] \in \mathbb{C}^{2R \times 2R}$ is the autocorrelation matrix of $\mathbf{z}(k)$ which, accounting for (16), and invoking assumptions A1 and A2), is given by

$$\mathbf{R}_{\mathbf{zz}} = \sigma_b^2 \boldsymbol{\mathcal{H}} \boldsymbol{\mathcal{H}}^H + \sigma_v^2 \mathbf{I}_{2R}. \tag{19}$$

Reasoning as in precedence for the L-MMSE multiuser detector, it is readily seen that, if $\boldsymbol{\mathcal{H}}$ is full-column rank,[5] the WL-MMSE multiuser detector ends up to the WL-ZF one in the limit $\sigma_v^2/\sigma_b^2 \to 0$.

Henceforth, similarly to the condition $\mathrm{rank}(\boldsymbol{\mathcal{G}}) = J$ for L-MUD, the full-column rank property of $\boldsymbol{\mathcal{H}}$ not only assures the existence of WL-ZF solutions, but also allows the WL-MMSE multiuser detector to satisfactorily work in the high SNR region. Such a condition, i.e., $\mathrm{rank}(\boldsymbol{\mathcal{H}}) = J$, is implicitly or explicitly advocated in the WL literature regarding DS-CDMA systems [14]–[18], [25], without providing conditions guaranteeing it. The full column rank property of $\boldsymbol{\mathcal{H}}$ is thoroughly studied in Section IV-A, with reference to both CP-and ZP-based systems. In particular, by taking advantage of the results derived in Section III, we will show that, if the user codes are judiciously designed, the condition $\mathrm{rank}(\boldsymbol{\mathcal{H}}) = J$ can also be guaranteed when the number of users exceeds the number of subcarriers, regardless of the underlying frequency-selective channel.

A. Rank Characterization of $\boldsymbol{\mathcal{H}}$ and Universal Code Design for WL-ZF-MUD

From a unified perspective, observe that $\mathrm{rank}(\boldsymbol{\mathcal{H}}) = J$ iff the null spaces of the matrices $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{G}}^*$ intersect only trivially, that is, $\mathcal{N}(\boldsymbol{\mathcal{G}}) \cap \mathcal{N}(\boldsymbol{\mathcal{G}}^*) = \{\mathbf{0}_J\}$. It can be easily verified that, if $\boldsymbol{\mathcal{G}}$ is full-column rank, which necessarily requires that $J \leq N$ (underloaded systems), then this condition is trivially satisfied and, hence, the augmented matrix $\boldsymbol{\mathcal{H}}$ is full-column rank as well. Remarkably, the converse statement is not true, that is, $\boldsymbol{\mathcal{H}}$ may be full-column rank even in overloaded MC-CDMA systems, i.e., when the number J of users is larger then the number N of subcarriers and, thus, $\boldsymbol{\mathcal{G}}$ is inherently rank-deficient. In the latter case, the code vectors $\{\mathbf{c}_j\}_{j=1}^J$ *cannot* be linearly independent, thus giving $\mathrm{rank}(\boldsymbol{\mathcal{G}}) \leq N$, which in its turn implies that the dimension of the subspace $\mathcal{N}(\boldsymbol{\mathcal{G}})$ is nonnull and is equal to $J - \mathrm{rank}(\boldsymbol{\mathcal{G}})$. In order to give more insights about this point, for the sake of clarity, it is convenient to consider ZP- and CP-based systems separately.

[5]More generally, when $\boldsymbol{\mathcal{H}}$ is possibly rank-deficient, then $\lim_{\sigma_v^2/\sigma_b^2 \to 0} \bar{\mathbf{f}}_{\mathrm{WL-MMSE},j} = (\boldsymbol{\mathcal{H}}^H)^\dagger \mathbf{e}_j \triangleq \bar{\mathbf{f}}_{\mathrm{WL-LS},j}$, i.e., the WL-MMSE multiuser detector ends up to the minimal-norm least-squares solution [33] of $\boldsymbol{\mathcal{H}}^H \bar{\mathbf{f}}_j = \mathbf{e}_j$ [note that, when $\boldsymbol{\mathcal{H}}$ is full-column rank, one has $\bar{\mathbf{f}}_{\mathrm{WL-LS},j} = \bar{\mathbf{f}}_{\mathrm{WL-ZF},j}$ from (17)].

1) ZP-Based Downlink: Let us consider a ZP-based system [see (5)], wherein $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}_{\mathrm{zp}} = \boldsymbol{\Theta}_{\mathrm{zp}} \mathbf{W}_{\mathrm{IDFT}} \mathbf{C}$. In this case, the augmented channel matrix $\boldsymbol{\mathcal{H}}$ assumes the form

$$\begin{aligned}\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}_{\mathrm{zp}} &= \begin{bmatrix} \boldsymbol{\mathcal{G}}_{\mathrm{zp}} \\ \boldsymbol{\mathcal{G}}_{\mathrm{zp}}^* \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \boldsymbol{\Theta}_{\mathrm{zp}} \mathbf{W}_{\mathrm{IDFT}} & \mathbf{O}_{P \times N} \\ \mathbf{O}_{P \times N} & \boldsymbol{\Theta}_{\mathrm{zp}}^* \mathbf{W}_{\mathrm{IDFT}}^* \end{bmatrix}}_{\overline{\boldsymbol{\Xi}}_{\mathrm{zp}} \in \mathbb{C}^{2P \times 2N}} \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}^* \end{bmatrix}}_{\overline{\mathbf{C}} \in \mathbb{C}^{2N \times J}} \\ &= \overline{\boldsymbol{\Xi}}_{\mathrm{zp}} \overline{\mathbf{C}}. \end{aligned} \tag{20}$$

It can be readily seen [34] that $\mathrm{rank}(\overline{\boldsymbol{\Xi}}_{\mathrm{zp}}) = \mathrm{rank}(\boldsymbol{\Theta}_{\mathrm{zp}} \mathbf{W}_{\mathrm{IDFT}}) + \mathrm{rank}(\boldsymbol{\Theta}_{\mathrm{zp}}^* \mathbf{W}_{\mathrm{IDFT}}^*) = 2N$ and, consequently, it follows that $\mathrm{rank}(\boldsymbol{\mathcal{H}}_{\mathrm{zp}}) = \mathrm{rank}(\overline{\mathbf{C}})$. In other words, let $\overline{\mathbf{c}}_j \triangleq [\mathbf{c}_j^T, \mathbf{c}_j^H]^T \in \mathbb{C}^{2N}$ define the *augmented* code vector of the jth user, for $j \in \{1, 2, \ldots, J\}$, the matrix $\boldsymbol{\mathcal{H}}_{\mathrm{zp}}$ is full-column rank iff the code vectors $\overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2, \ldots, \overline{\mathbf{c}}_J$ are linearly independent. In other words, a necessary and sufficient condition guaranteeing the existence of WL-ZF solutions for ZP-based system is that the augmented code matrix $\overline{\mathbf{C}}$ is full-column rank. It is worthwhile to observe that the augmented code vectors $\{\overline{\mathbf{c}}_j\}_{j=1}^J$ can be linearly independent even if the code vectors $\mathbf{c}_j$ are linearly dependent, which surely happens when $J > N$. In this regard, we provide the following lemma.

Lemma 1 (Rank Characterization of $\overline{\mathbf{C}}$): If $J \leq 2N$, then the augmented frequency-domain code matrix $\overline{\mathbf{C}}$ is full-column rank iff there are no conjugate pairs of nonzero vectors belonging to $\mathcal{N}(\mathbf{C})$.

Proof: Preliminarily, observe that $\mathrm{rank}(\overline{\mathbf{C}}) = J$ iff the null spaces of $\mathbf{C}$ and $\mathbf{C}^*$ intersect only trivially, that is, $\mathcal{N}(\mathbf{C}) \cap \mathcal{N}(\mathbf{C}^*) = \{\mathbf{0}_J\}$. An arbitrary *nonzero* vector $\boldsymbol{\alpha} \in \mathbb{C}^J$ belongs to $\mathcal{N}(\mathbf{C})$ iff $\mathbf{C}\boldsymbol{\alpha} = \mathbf{0}_N$, from which, by conjugating, one obtains $\mathbf{C}^* \boldsymbol{\alpha}^* = \mathbf{0}_N$. The last two systems of equations show that $\boldsymbol{\alpha} \in \mathcal{N}(\mathbf{C})$ iff $\boldsymbol{\alpha}^* \in \mathcal{N}(\mathbf{C}^*)$. Consequently, an arbitrary vector $\boldsymbol{\alpha} \neq \mathbf{0}_J$ belongs to $\mathcal{N}(\mathbf{C}) \cap \mathcal{N}(\mathbf{C}^*)$ iff there exists a nonzero vector $\boldsymbol{\beta} \in \mathbb{C}^J$ belonging to $\mathcal{N}(\mathbf{C})$ such that $\boldsymbol{\beta}^* = \boldsymbol{\alpha}$. ■

In underloaded scenarios, wherein the code vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_J$ can be linearly independent, it follows that $\mathcal{N}(\mathbf{C}) = \{\mathbf{0}_J\}$ and, thus, the augmented matrix $\overline{\mathbf{C}}$ turns out to be full-column rank, too. Therefore, from now on, we focus attention on the more interesting overloaded environments, wherein $N > J \leq 2N$. In this case, $\mathbf{C}$ is a wide matrix and, assuming without loss of generality that its first N column $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_N$ are linearly independent, its remaining $J - N$ columns $\mathbf{c}_{N+1}, \mathbf{c}_{N+2}, \ldots, \mathbf{c}_J$ can be expressed as a linear combination of the first N ones, thus, obtaining the following decomposition:

$$\mathbf{C} = [\mathbf{C}_{\mathrm{left}}\ \mathbf{C}_{\mathrm{left}}\, \boldsymbol{\Pi}] = \mathbf{C}_{\mathrm{left}} [\mathbf{I}_N\ \boldsymbol{\Pi}] \tag{21}$$

where $\mathbf{C}_{\mathrm{left}} \triangleq [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_N] \in \mathbb{C}^{N \times N}$ is nonsingular and $\boldsymbol{\Pi} \in \mathbb{C}^{N \times (J-N)}$ is a tall matrix. Due to nonsingularity of $\mathbf{C}_{\mathrm{left}}$, it follows that $\mathcal{N}(\mathbf{C}) = \mathcal{N}([\mathbf{I}_N \; \mathbf{\Pi}])$. Furthermore, it can be verified that the general form of two vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{C}^J$ belonging to $\mathcal{N}([\mathbf{I}_N \; \mathbf{\Pi}])$ and, thus, to $\mathcal{N}(\mathbf{C})$, is given by

$$\boldsymbol{\alpha}_1 = \begin{bmatrix} -\mathbf{\Pi} \\ \mathbf{I}_{J-N} \end{bmatrix} \boldsymbol{\vartheta}_1 \quad \text{and} \quad \boldsymbol{\alpha}_2 = \begin{bmatrix} -\mathbf{\Pi} \\ \mathbf{I}_{J-N} \end{bmatrix} \boldsymbol{\vartheta}_2 \tag{22}$$

with arbitrary $\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2 \in \mathbb{C}^{J-N}$. By virtue of Lemma 1, the augmented code matrix $\overline{\mathbf{C}}$ is not full-column rank iff there exist at least two nonzero vectors $\boldsymbol{\vartheta}_1$ and $\boldsymbol{\vartheta}_2$ such that $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2^*$, which amounts to $\boldsymbol{\vartheta}_1 = \boldsymbol{\vartheta}_2^*$ and $(\mathbf{\Pi} - \mathbf{\Pi}^*)\boldsymbol{\vartheta}_1 = \mathbf{0}_N$. In its turn this second equation can be equivalently written as $\text{Im}\{\mathbf{\Pi}\}\boldsymbol{\vartheta}_1 = \mathbf{0}_N$. Therefore, if the imaginary part of $\mathbf{\Pi}$ is full-column rank, then $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2^*$ is satisfied iff $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \mathbf{0}_J$ which, accounting for Lemma 1, assures that $\text{rank}(\overline{\mathbf{C}}) = J$. Summarizing this result, we can state the following universal code design strategy for a ZP-based overloaded system.

Condition $\overline{\mathrm{D}}_{\text{zp}}$ *(Universal Code Design for WL-ZF-MUD in ZP-MC-CDMA):* Let $N < J \leq 2N$ and $\mathbf{C}_{\text{left}} \triangleq [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_N] \in \mathbb{C}^{N \times N}$ be nonsingular, the code matrix has the form $\mathbf{C} = \mathbf{C}_{\text{left}}[\mathbf{I}_N \; \mathbf{\Pi}]$, where $\mathbf{\Pi} \in \mathbb{C}^{N \times (J-N)}$ is a tall matrix, whose imaginary part $\text{Im}\{\mathbf{\Pi}\}$ is full-column rank.

Some interesting remarks regarding fulfillment of condition $\overline{\mathrm{D}}_{\text{zp}}$ can be drawn at this point.

Remark 5: To begin with, observe that the code design $\overline{\mathrm{D}}_{\text{zp}}$, which represents a necessary and sufficient condition in order to guarantee $\text{rank}(\overline{\mathbf{C}}) = J$, is universal, in the sense that it allows $\boldsymbol{\mathcal{H}}_{\text{zp}}$ to be full-column rank for *any* FIR channel of order $L < L_{\text{p}}$. If this universal code constraint is fulfilled, then channel-irrespective WL-ZF-MUD is guaranteed up to $2N$ users, which is exactly the double of the number of users that can be managed in a ZP-based system employing L-ZF-MUD.

Remark 6: If the spreading codes are real-valued, i.e., $\mathbf{C}^* = \mathbf{C}$, the matrix $\mathbf{\Pi}$ is real-valued as well, i.e., $\text{Im}\{\mathbf{\Pi}\} = \mathbf{O}_{N \times (J-N)}$ and, consequently, condition $\overline{\mathrm{D}}_{\text{zp}}$ is not satisfied. Thus, employing real-valued code vectors (e.g., WH spreading) implies necessarily that, similarly to L-ZF-MUD, the existence of WL-ZF solutions can be guaranteed only in underloaded MC-CDMA systems. On the other hand, if complex-valued code vectors are employed, then $\overline{\mathbf{C}}$ can be full-column rank even in overloaded systems, where $\mathbf{C}$ is not full-column rank. Observe that the use of complex spreading has been originally proposed also for DS-CDMA systems, either to reduce the peak-to-average power ratio of the transmitted signal [40] and, thus, improving RF power amplifier efficiency, or to synthesize sequences that exhibit better autocorrelation properties than WH codes [41].

Remark 7: Although they are complex-valued, the VM code vectors given by (13) do not satisfy $\overline{\mathrm{D}}_{\text{zp}}$ when $N < J \leq 2N$: indeed, it is easily shown that, in this case, the following decomposition holds $\mathbf{C} = \mathbf{W}_{\text{DFT}}[\mathbf{I}_N \; \mathbf{J}]$, where $\mathbf{J} \triangleq [\mathbf{1}_1, \mathbf{1}_2, \ldots, \mathbf{1}_{J-N}] \in \mathbb{R}^{N \times (J-N)}$ is real-valued and, in this case, $\text{Im}\{\mathbf{\Pi}\} = \text{Im}\{\mathbf{J}\} = \mathbf{O}_{N \times (J-N)}$ is rank-deficient. Hence, the VM codes (13) do not ensure channel-independent WL-ZF-MUD in an overloaded ZP-based downlink.

Besides allowing one to readily check whether a given set of spreading sequences assures the existence of WL-ZF solutions for any FIR channel of order $L < L_{\text{p}}$, condition $\overline{\mathrm{D}}_{\text{zp}}$ provides a direct procedure to build universal codes for ZP-based overloaded systems. Among several options that can be pursued, we devise here a simple universal code design relying on WH spreading. Specifically, let $\boldsymbol{\mathcal{W}}_N \in \mathbb{R}^{N \times N}$ denote the common Hadamard matrix of order N: in underloaded scenarios, i.e., when $J \leq N$, one can choose the spreading vectors $\{\mathbf{c}_j\}_{j=1}^J$ as the columns of $(1/\sqrt{N})\boldsymbol{\mathcal{W}}_N$ (the normalization by $1/\sqrt{N}$ assures that $\|\mathbf{c}_j\|^2 = 1$ for each user); on the other hand, in an overloaded downlink, wherein $N < J \leq 2N$, the code matrix $\mathbf{C}$ can be chosen as follows

$$\mathbf{C} = \frac{1}{\sqrt{N}}(\boldsymbol{\mathcal{W}}_N[\mathbf{I}_N \; i\mathbf{J}]) \tag{23}$$

which, as it is immediately seen, satisfies condition $\overline{\mathrm{D}}_{\text{zp}}$. In this way, the spreading vectors of the first N users have elements confined to the two values $\{\pm 1/\sqrt{N}\}$, whereas the entries of the code vectors of the remaining users take on the two values $\{\pm i/\sqrt{N}\}$. In conclusion, we can state that the adoption of the code matrix (23), which comes from a simple modification of the conventional WH spreading technique, guarantees WL-ZF-MUD in both underloaded and overloaded ZP-based downlink, for any FIR channel of order $L < L_{\text{p}}$.

2) CP-Based Downlink: Let us consider a CP-based system [see (4)], wherein $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}_{\text{cp}}$ can be equivalently expressed as $\boldsymbol{\mathcal{G}}_{\text{cp}} = \mathbf{W}_{\text{IDFT}}\mathbf{\Gamma}_{\text{cp}}\mathbf{C}$ (see Section III-A). In this case, one has

$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}_{\text{cp}} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_{\text{cp}} \\ \boldsymbol{\mathcal{G}}_{\text{cp}}^* \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{W}_{\text{IDFT}} & \mathbf{O}_{N \times N} \\ \mathbf{O}_{N \times N} & \mathbf{W}_{\text{IDFT}}^* \end{bmatrix}}_{\overline{\mathbf{W}}_{\text{IDFT}} \in \mathbb{C}^{2N \times 2N}} \cdot \underbrace{\begin{bmatrix} \mathbf{\Gamma}_{\text{cp}} & \mathbf{O}_{N \times N} \\ \mathbf{O}_{N \times N} & \mathbf{\Gamma}_{\text{cp}}^* \end{bmatrix}}_{\overline{\mathbf{\Gamma}}_{\text{cp}} \in \mathbb{C}^{2N \times 2N}} \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}^* \end{bmatrix}}_{\overline{\mathbf{C}} \in \mathbb{C}^{2N \times J}} = \overline{\mathbf{W}}_{\text{IDFT}}\overline{\mathbf{\Gamma}}_{\text{cp}}\overline{\mathbf{C}}. \tag{24}$$

Since $\text{rank}(\overline{\mathbf{W}}_{\text{IDFT}}) = \text{rank}(\mathbf{W}_{\text{IDFT}}) + \text{rank}(\mathbf{W}_{\text{IDFT}}^*) = 2N$, it results that $\text{rank}(\boldsymbol{\mathcal{H}}_{\text{cp}}) = \text{rank}(\overline{\mathbf{\Gamma}}_{\text{cp}}\overline{\mathbf{C}})$ and, hence, we can directly investigate the rank properties of $\overline{\mathbf{\Gamma}}_{\text{cp}}\overline{\mathbf{C}}$. As a first remark, observe that, in order for $\overline{\mathbf{\Gamma}}_{\text{cp}}\overline{\mathbf{C}}$ to be full-column rank, the matrix $\overline{\mathbf{C}}$ must necessarily be full-column rank, i.e., $J \leq 2N$ and $\text{rank}(\overline{\mathbf{C}}) = J$. Therefore, differently from the ZP case, linear independence of the augmented code vector $\overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2, \ldots, \overline{\mathbf{c}}_J$ is a necessary but not sufficient condition in order to have $\text{rank}(\boldsymbol{\mathcal{H}}_{\text{cp}}) = J$. Consequently, to allow $\boldsymbol{\mathcal{H}}_{\text{cp}}$ to be full-column rank even in overloaded scenarios, as a first constraint on the user codes, we have to impose that the matrix $\mathbf{C}$ be synthesized according to $\overline{\mathrm{D}}_{\text{zp}}$, which represents a necessary and sufficient condition in order to have $\text{rank}(\overline{\mathbf{C}}) = J$, when $N < J \leq 2N$. This implies that any spreading technique, which enables channel-irrespective WL perfect symbol recovery for a CP-based downlink, can also be employed for the same purpose in a ZP-based system. The full-column rank property of the matrix $\boldsymbol{\mathcal{H}}_{\text{cp}}$ is characterized by the following Theorem.

Theorem 2 (Rank Characterization of $\boldsymbol{\mathcal{H}}_{\text{cp}}$*):* If $\overline{\mathbf{C}}$ is full-column rank and the channel transfer function $G(z)$ has $0 \leq M_z \leq L$ distinct zeros on the subcarriers $z_{m_1} = e^{i(2\pi/N)m_1}, z_{m_2} = e^{i(2\pi/N)m_2}, \ldots, z_{m_{M_z}} = e^{i(2\pi/N)m_{M_z}}$, with $m_1 \neq m_2 \neq \cdots \neq m_{M_z} \in \{0, 1, \ldots, N-1\}$, then the augmented channel matrix $\boldsymbol{\mathcal{H}}_{\text{cp}}$ is full-column rank iff $[\overline{\mathbf{C}}, \overline{\mathbf{S}}_z] \in \mathbb{C}^{2N \times (J+2M_z)}$ is full-column rank, where $\overline{\mathbf{S}}_z \triangleq \text{diag}[\mathbf{S}_z, \mathbf{S}_z] \in \mathbb{R}^{2N \times 2M_z}$ is full-column rank and $\mathbf{S}_z \in \mathbb{R}^{N \times M_z}$ has been previously defined in Theorem 1. ■

Proof: The proof is similar in spirit with that of Theorem 1 and, thus, is omitted. ■

Theorem 2 suggests the following two additional remarks.

Remark 8: As a first consequence, if the channel transfer function $G(z)$ has no zeros on the subcarriers $\{z_m\}_{m=0}^{N-1}$, i.e., $M_z = 0$, then, similarly to a ZP-based system, the linear independence of the augmented code vectors $\overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2, \ldots, \overline{\mathbf{c}}_J$ becomes a necessary *and* sufficient condition for the existence of WL-ZF solutions in a CP-based downlink. In this case, a CP-based downlink can support up to $2N$ active users, which is equal to the system capacity of a ZP-based downlink employing WL-ZF-MUD. Instead, in the presence of channel zeros on some subcarriers, $\boldsymbol{\mathcal{H}}_{\text{cp}}$ can still be full-column rank. However, in this case, provided that $\overline{\mathbf{C}}$ is full-column rank, the existence of WL-ZF solutions explicitly depends on the channel-zero configuration.

Remark 9: Most importantly, unlike the condition $\text{rank}([\mathbf{C}, \mathbf{S}_z]) = J + M_z$ of Theorem 1 (see also Remark 3), the condition $\text{rank}([\overline{\mathbf{C}}, \overline{\mathbf{S}}_z]) = J + 2M_z$ can be satisfied even when the number of users is larger than the number of subcarriers. Specifically, $\text{rank}([\overline{\mathbf{C}}, \overline{\mathbf{S}}_z]) = J + 2M_z$ necessarily requires that $2N \geq J + 2M_z$, that is, the number J of active users must not be larger than $2(N - M_z)$, with $0 < M_z \leq L < L_{\text{p}} \ll N$. Hence, similarly to a ZP-based system, WL-ZF-MUD allows a CP-based downlink to support a number of users that is exactly the double of the number of users that can be accommodated when L-ZF-MUD is employed. However, in the latter case, the allowable number of users is decremented by *two* units for any additional zero on the subcarriers and is smaller than $2N$, which represents the system capacity of a ZP-based system employing WL-ZF-MUD. In the worst case, when all the channel zeros are located at the subcarriers, the maximum number of allowable users in a CP-based downlink is $2(N - L)$.

Similarly to Theorem 1, the most important implication of Theorem 2 regards the fact that it enlightens how to single out universal code designs, which assure that $\boldsymbol{\mathcal{H}}_{\text{cp}}$ be full-column rank for any possible configuration of the channel zeros. With this goal in mind, paralleling the arguments that led to condition D_{cp} in Section III-A, the following code design represents a necessary and sufficient condition ensuring that $\boldsymbol{\mathcal{H}}_{\text{cp}}$ is full-column rank for any possible configuration of the channel zeros.

Condition $\overline{\text{D}}_{\text{cp}}$ *(Universal Code Design for WL-ZF-MUD in CP-MC-CDMA):* Define the full-column rank matrix $\overline{\mathbf{S}}_{\text{univ}} \triangleq \text{diag}[\mathbf{S}_{\text{univ}}, \mathbf{S}_{\text{univ}}] \in \mathbb{R}^{2N \times 2L}$, where $\mathbf{S}_{\text{univ}} \in \mathbb{R}^{N \times L}$ has been previously defined in condition D_{cp}, then, $\forall \{m_1, m_2, \ldots, m_L\} \subset \{0, 1, \ldots, N-1\}$

$$\text{rank}([\overline{\mathbf{C}}, \overline{\mathbf{S}}_{\text{univ}}]) = J + 2L \quad \text{or equivalently}$$

$$\text{rank}\left[\left(\mathbf{I}_{2N} - \overline{\mathbf{S}}_{\text{univ}} \overline{\mathbf{S}}_{\text{univ}}^T\right) \overline{\mathbf{C}}\right] = J. \quad (25)$$

The price to pay for imposing that the matrix $[\overline{\mathbf{C}}, \overline{\mathbf{S}}_{\text{univ}}] \in \mathbb{C}^{2N \times (J+2L)}$ be full-column rank is a reduction of the system capacity (see Remark 9): the universal code design $\overline{\text{D}}_{\text{cp}}$ can be devised for a maximum number of $2(N - L)$ users. It should be observed that $\overline{\text{D}}_{\text{cp}}$ is stronger than condition $\overline{\text{D}}_{\text{zp}}$: indeed, $\overline{\text{D}}_{\text{cp}}$ necessarily requires that $\text{rank}(\overline{\mathbf{C}}) = J$; on the other hand, $\text{rank}(\overline{\mathbf{C}}) = J$ is not sufficient to assure fulfillment of $\overline{\text{D}}_{\text{cp}}$. On the other hand, it is noteworthy that, if condition D_{cp} is satisfied, which is possible as long as $J \leq N - L$, then $\overline{\text{D}}_{\text{cp}}$ is surely fulfilled, too. Therefore, by imposing the unique constraint that the N parameters $\{\rho_\ell\}_{\ell=0}^{N-1}$ be distinct, the code vectors (12) guarantee, up to $N - L$ users, the existence of universal WL-ZF solutions. However, in its present form, condition $\overline{\text{D}}_{\text{cp}}$ does not help us give a direct procedure for synthesizing universal spreading codes when $N - L < J \leq 2(N - L)$ and, thus, some further development is needed. Relying on the fact that the matrix $(\mathbf{I}_{2N} - \overline{\mathbf{S}}_{\text{univ}} \overline{\mathbf{S}}_{\text{univ}}^T)\overline{\mathbf{C}} \in \mathbb{C}^{2N \times J}$ is obtained from $\overline{\mathbf{C}}$ by setting to zero all the entries of its $2L$ rows located in the positions $m_1 + 1, m_2 + 1, \ldots, m_L + 1, m_1 + N + 1, m_2 + N + 1, \ldots, m_L + N + 1$, with reference to the specific case wherein $N - L < J \leq 2(N - L)$, we are now able to state the following equivalent reformulation of condition $\overline{\text{D}}_{\text{cp}}$.

Condition $\overline{\text{D}}_{\text{cp}}$ *[Reformulation When* $N - L < J \leq 2(N - L)$*]:* Let $\boldsymbol{\omega}_\ell^T \triangleq [c_1^{(\ell)}, c_2^{(\ell)}, \ldots, c_J^{(\ell)}] \in \mathbb{C}^{1 \times J}$ denote the $(\ell + 1)$th row of $\mathbf{C}$, with $\ell \in \{0, 1, \ldots, N - 1\}$; when $N - L < J \leq 2(N - L)$, for any subset of distinct indices $\{\ell_1, \ell_2, \ldots, \ell_{N-L}\} \subset \{0, 1, \ldots, N - 1\}$, there exists J linearly independent vectors from the total set $\boldsymbol{\omega}_{\ell_1}, \boldsymbol{\omega}_{\ell_2}, \ldots, \boldsymbol{\omega}_{\ell_{N-L}}, \boldsymbol{\omega}_{\ell_1}^*, \boldsymbol{\omega}_{\ell_2}^*, \ldots, \boldsymbol{\omega}_{\ell_{N-L}}^*$.

Reformulation of condition $\overline{\text{D}}_{\text{cp}}$ allows one to readily check out that the code vectors (12) can still fulfill $\overline{\text{D}}_{\text{cp}}$ when $N - L < J \leq 2(N - L)$, provided that, in addition to $\rho_0 \neq \rho_1 \neq \cdots \neq \rho_{N-1}$, further constraints on the parameters $\{\rho_\ell\}_{\ell=0}^{N-1}$ are imposed. More precisely, relying on the properties of Vandermonde vectors [34], it is not difficult to prove that condition $\overline{\text{D}}_{\text{cp}}$ is surely satisfied if, besides requiring that the parameters $\{\rho_\ell\}_{\ell=0}^{N-1}$ be distinct, one additionally imposes that $\rho_{\ell_1} \neq \rho_{\ell_2}^*$, $\forall \ell_1, \ell_2 \in \{0, 1, \ldots, N - 1\}$, which means that the number $\{\rho_\ell\}_{\ell=0}^{N-1}$ must be complex-valued and cannot be pairwise conjugate. Additionally, it can be immediately inferred that the VM codes (13) cannot satisfy the code design $\overline{\text{D}}_{\text{cp}}$ since, in this case, it turns out that $\rho_\ell = \rho_{N-\ell}^*$, $\forall \ell \in \{0, 1, \ldots, N - 1\}$. Furthermore, it can be verified by direct inspection that the code matrix given by (23) does not satisfy condition $\overline{\text{D}}_{\text{cp}}$ (see also Section V) and, thus, contrary to the ZP case, such a spreading technique does not guarantee the existence of universal WL-ZF solutions in CP-based systems. At this point, it is paramount to develop a family of codes fulfilling condition $\overline{\text{D}}_{\text{cp}}$. To this aim, we restrict our attention to the spreading vectors (12) and, in particular, we start from the N-point DFT codes (13), whereby $\rho_\ell = e^{-i(2\pi/N)\ell}, \forall \ell \in \{0, 1, \ldots, N - 1\}$. To obtain a set of N complex-valued parameters $\{\rho_\ell\}_{\ell=0}^{N-1}$ equispaced on the unit circle, which are not pairwise conjugate, it is sufficient to introduce a suitable rotation by setting $\rho_\ell = e^{-i((2\pi/N)\ell - \theta)}$, $\forall \ell \in \{0, 1, \ldots, N - 1\}$ and $\theta \in (0, 2\pi)$, thus getting the code vectors

$$\mathbf{c}_j = \frac{1}{\sqrt{N}} \left[e^{-i(-\theta)j}, e^{-i\left(\frac{2\pi}{N} - \theta\right)j}, \ldots, e^{-i\left(\frac{2\pi}{N}(N-1) - \theta\right)j}\right]^T \quad (26)$$

$\forall j \in \{1, 2, \ldots, J\}$, where, in order to fulfill the constraint $\rho_{\ell_1} \neq \rho_{\ell_2}^*$, $\forall \ell_1, \ell_2 \in \{0, 1, \ldots, N - 1\}$, the angle rotation θ must obey the following condition: $\theta \neq (\pi/N)(\ell_1 + \ell_2) + h\pi$,

$\forall \ell_1, \ell_2 \in \{0, 1, \ldots, N-1\}$ and $\forall h \in \mathbb{Z}$. Note that the spreading vectors (26) differ from those in (13) only for the multiplicative scalar $e^{-i(-\theta)j}$. The code vectors (26) satisfy the condition $\overline{\mathrm{D}}_{\mathrm{cp}}$ and, hence, they ensure universal WL perfect symbol recovery not only when $J \leq N - L$, but also when $N - L < J \leq 2(N - L)$, in both CP-and ZP-based systems. Finally, observe that, when L is replaced with L_{p}, universal WL-ZF-MUD is still possible in a CP-based system, with the difference that perfect symbol recovery can be guaranteed to at most $2(N - L_{\mathrm{p}})$ users, whose number, although does not depend on the channel order, is, however, smaller than $2(N - L)$.

V. Numerical Performance Analysis

To corroborate the theoretical results provided up to this point, we resort to Monte Carlo computer simulations in this section. Specifically, we consider that, without loss of generality, the desired user is the first one $(j = 1)$ and, moreover, we assume that $\mathbf{g}$ is exactly known at the receiver.

In all the experiments, the following simulation setting is adopted. The CP- and ZP-based MC-CDMA systems employ $N = 16$ subcarriers, with $L_{\mathrm{p}} = 4$ and OQPSK improper symbol modulation. Both systems use four different frequency-domain spreading sequences: the common WH spreading codes; the VM spreading vectors given by (13); the complex-valued WH (CWH) code vectors given by (23); the rotated VM (RVM) code vectors given by (26), with $\theta = \pi/32$. The baseband discrete-time multipath channel $\{g(\ell)\}_{\ell=0}^{L}$ is a FIR filter of order $L = 3$, whose transfer function is given by

$$G(z) = (1 - \zeta_1 z^{-1})(1 - \zeta_2 z^{-1})(1 - \zeta_3 z^{-1}) \tag{27}$$

where the group $(\zeta_1, \zeta_2, \zeta_3)$ of its three zeros assumes a different configuration in each Monte Carlo run. During the first 16 runs, we set $\zeta_1 = e^{i(2\pi/N)m_1}$ (one zero on the subcarriers), where, in each run, m_1 takes on a different value in $\{0, 1, \ldots, N-1\}$, whereas the magnitudes and phases of ζ_2 and ζ_3, which are modeled as mutually independent random variables uniformly distributed over the intervals $(0, 2)$ and $(0, 2\pi)$, respectively, are randomly and independently generated from run to run. During the subsequent $\binom{16}{2} = 120$ runs we set $\zeta_1 = e^{i(2\pi/N)m_1}$ and $\zeta_2 = e^{i(2\pi/N)m_2}$ (two zeros on the subcarriers), where, in each run, m_1 and m_2 take on a different value in $\{0, 1, \ldots, N-1\}$, with $m_1 \neq m_2$, whereas the magnitude and phase of ζ_3, which are modeled as mutually independent random variables uniformly distributed over the intervals $(0, 2)$ and $(0, 2\pi)$, respectively, are randomly and independently generated from run to run. During the last $\binom{16}{3} = 560$ runs we set $\zeta_1 = e^{i(2\pi/N)m_1}$, $\zeta_2 = e^{i(2\pi/N)m_2}$ and $\zeta_3 = e^{i(2\pi/N)m_3}$ (three zeros on the subcarriers), where, in each run, m_1, m_2, and m_3 take on a different value in $\{0, 1, \ldots, N-1\}$, with $m_1 \neq m_2 \neq m_3$. In this way, one obtains $16 + 120 + 560 = 696$ independent channel realizations. According to assumption A2), the entries of the noise vector $\mathbf{v}(k)$ [see (6)] are modeled as zero-mean i.i.d. complex circular Gaussian random variables, with variance σ_v^2, and the SNR of the desired user is defined as $\mathrm{SNR} \triangleq (\sigma_b^2 \|\mathbf{c}_1\|^2)/\sigma_v^2$ (since $\|\mathbf{c}_j\|^2 = 1, \forall j \in \{1, 2, \ldots, J\}$, all the users undergo the same SNR).

For both CP- and ZP-based systems, employing the aforementioned four different spreading sequences, we carried out a comparative performance study of the L-ZF, L-MMSE, WL-ZF, and WL-MMSE detectors.[6] At first sight, it seems that the synthesis of the L-ZF detector given by (7), which does not depend on the statistics of the received data, requires knowledge of the spreading codes of all the active users, which is an unreasonable requirement in the downlink. However, following the same lines of [7], this problem can be circumvented by implementing the L-ZF detector by means of the following SOS-based *subspace* representation

$$\mathbf{f}_{\mathrm{L-ZF},1} = \mathbf{V}_s \left(\mathbf{\Lambda}_s - \sigma_v^2 \mathbf{I}_J\right)^{-1} \mathbf{V}_s^H \mathbf{\Upsilon}_1 \mathbf{g} \tag{28}$$

where $\mathbf{V}_s \in \mathbb{C}^{R \times J}$ collects the eigenvectors associated with the J largest eigenvalues of $\mathbf{R}_{\mathbf{rr}}$ (arranged in descending order), which represents the diagonal entries of $\mathbf{\Lambda}_s \triangleq \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_J] \in \mathbb{R}^{J \times J}$, whereas: for a CP-based system $(R = N)$, $\mathbf{\Upsilon}_1 = \mathbf{\Upsilon}_{\mathrm{cp},1} \triangleq \tilde{\mathbf{\Phi}}_{\mathrm{cp},1} \mathbf{\Omega}_{\mathrm{cp}} \in \mathbb{C}^{N \times L_{\mathrm{p}}}$ is a known full-column rank matrix, with $\tilde{\mathbf{\Phi}}_{\mathrm{cp},1} \in \mathbb{C}^{N \times N}$ being a nonsingular circulant matrix, whose first column is given by $\tilde{\mathbf{c}}_1 \triangleq \mathbf{W}_{\mathrm{IDFT}} \mathbf{c}_1 \in \mathbb{C}^N$; for a ZP-based system $(R = P)$, $\mathbf{\Upsilon}_1 = \mathbf{\Upsilon}_{\mathrm{zp},1} \triangleq \tilde{\mathbf{\Phi}}_{\mathrm{zp},1} \mathbf{\Omega}_{\mathrm{zp}} \in \mathbb{C}^{P \times L_{\mathrm{p}}}$, where $\tilde{\mathbf{\Phi}}_{\mathrm{zp},1} \in \mathbb{C}^{P \times P}$ is a known lower triangular Toeplitz [34] matrix having as first column $[\tilde{\mathbf{c}}_1^T, 0, \ldots, 0]^T$ and as first row $[\tilde{c}_1^{(0)}, 0, \ldots, 0]$. In the subspace-based form (28), apart from $\mathbf{g}$ and the eigenstructure of $\mathbf{R}_{\mathbf{rr}}$ (which can be consistently estimated from the received data), the synthesis of the L-ZF detector requires only knowledge of the desired code vector $\mathbf{c}_1$. For a fair comparison, we implemented the subspace-based version of the L-MMSE detector defined in (8), which can be expressed [7], [42] as

$$\mathbf{f}_{\mathrm{L-MMSE},1} = \mathbf{V}_s \mathbf{\Lambda}_s^{-1} \mathbf{V}_s^H \mathbf{\Upsilon}_1 \mathbf{g}. \tag{29}$$

The derivations reported in [7] and [42], which exclusively consider linear receiving structure, can be suitably extended to obtain the subspace versions of the WL-ZF and WL-MMSE detectors given by (17) and (18), respectively, thus, obtaining (for the sake of brevity, we omit the mathematical details)

$$\mathbf{f}_{\mathrm{WL-ZF},1} = \mathbf{U}_s \left(\mathbf{\Sigma}_s - \sigma_v^2 \mathbf{I}_J\right)^{-1} \mathbf{U}_s^H \begin{bmatrix} \mathbf{\Upsilon}_1 \mathbf{g} \\ \mathbf{\Upsilon}_1^* \mathbf{g}^* \end{bmatrix} \tag{30}$$

$$\mathbf{f}_{\mathrm{WL-MMSE},1} = \mathbf{U}_s \mathbf{\Sigma}_s^{-1} \mathbf{U}_s^H \begin{bmatrix} \mathbf{\Upsilon}_1 \mathbf{g} \\ \mathbf{\Upsilon}_1^* \mathbf{g}^* \end{bmatrix} \tag{31}$$

where $\mathbf{U}_s \in \mathbb{C}^{2R \times J}$ collects the eigenvectors associated with the J largest eigenvalues $\mu_1, \mu_2, \ldots, \mu_J$ of $\mathbf{R}_{\mathbf{zz}}$ (arranged in descending order) and $\mathbf{\Sigma}_s \triangleq \mathrm{diag}[\mu_1, \mu_2, \ldots, \mu_J] \in \mathbb{R}^{J \times J}$. In all the experiments, sample estimates of the eigenvectors and eigenvalues (including the noise variance σ_v^2 needed for the synthesis of the ZF detectors) of $\mathbf{R}_{\mathbf{rr}}$ and $\mathbf{R}_{\mathbf{zz}}$ were obtained in batch-mode from the sample autocorrelation matrices $\hat{\mathbf{R}}_{\mathbf{rr}}$ and $\hat{\mathbf{R}}_{\mathbf{zz}}$, respectively, by using a data record of $K = 500$ symbols. Finally, as performance measure, we resorted to the average BER (ABER) at the output of the considered receivers:

[6]In the sequel, for notational convenience, a particular detector, which operates in a system employing a given set of spreading sequences, will be synthetically referred to through the acronym of the detector followed by the acronym of the code enclosed in round brackets; for example, the notation "L-ZF (WH)" means that the L-ZF detector is used at the receiver and, at the same time, WH spreading codes are employed at the transmitter.

after estimating the detector weight vectors on the basis of the given data record, for each of the 696 Monte Carlo run (wherein, besides the channel impulse response, independent sets of noise and data sequences were randomly generated), an independent record of $K_{\text{aber}} = 10^5$ symbols was considered to evaluate the ABER.

1) ABER Versus SNR: In the first group of experiments, we evaluated the performances of the considered receivers as a function of the SNR ranging from 0 to 20 dB.

In the first two experiments, we preliminarily studied the performances of the L-ZF and L-MMSE detectors: since linear receivers can work only when $J \leq N$, we considered in these experiments underloaded CP- and ZP-based systems, with $J = 10$ active users. In Fig. 1, we considered a CP-based system employing either WH or VM spreading codes.[7] In this case, it is apparent from Fig. 1 that the performances of both the "L-ZF (WH)" and "L-MMSE (WH)" detectors exhibit a marked floor in the high SNR region, which is the natural consequence of the fact that, for a CP-based downlink, WH spreading sequences do not ensure the existence of L-ZF solutions when the channel transfer function exhibits zeros located on the subcarriers. On the other hand, when VM codes are used, perfect symbol recovery in the absence of noise is guaranteed regardless of the channel zero locations. In fact, as it is shown in Fig. 1, the curves of both the "L-ZF (VM)" and "L-MMSE (VM)" detectors go down very quickly as the SNR increases, thus assuring a huge performance gain with respect to the "L-ZF (WH)" and "L-MMSE (WH)" receivers. The results of Fig. 2 were instead obtained by considering a ZP-based downlink. In this scenario, both WH and VM codes assure the existence of L-ZF solutions for any FIR channel of order $L < L_{\text{p}}$. Indeed, as it is apparent from Fig. 2, the performances of all the receivers under comparison rapidly improve for increasing values of the SNR. With regard to the L-ZF receivers, it is noteworthy that the "L-ZF (WH)" detector performs better than the "L-ZF (VM)" one: specifically, with respect to the "L-ZF (VM)" receiver, the "L-ZF (WH)" detector saves about 4 dB in transmitter power, for a target ABER of 10^{-4}. This means that, in comparison with VM spreading, WH codes lead to a reduced noise enhancement at the receiver output. Anyway, this performance gap is substantially halved if one brings the same comparison between the performances of the "L-MMSE (WH)" and "L-MMSE (VM)" detectors.

In the following two experiments, we investigated the performances of the WL-ZF and WL-MMSE detectors: since WL receivers can work even when $J > N$, we simulated in these experiments overloaded CP- and ZP-based systems with $J = 20$ active users. With reference to a CP-based system, results of Fig. 3 show that the "WL-ZF (WH)," "WL-MMSE (WH)," "WL-ZF (VM)," and "WL-MMSE (VM)" receivers do not work at all. As previously pointed out in Remarks 6 and 7, these catastrophic performances arise since, not only the WH spreading codes, but also the VM code vectors do not assure the full-column rank property of the augmented code matrix

[7]The results regarding CWH code vectors are not reported since, for underloaded systems, they end up to the WH spreading sequences; additionally, in the same scenario, we do not report the results concerning the RVM spreading vectors since they are very similar to those presented for the VM code vectors.

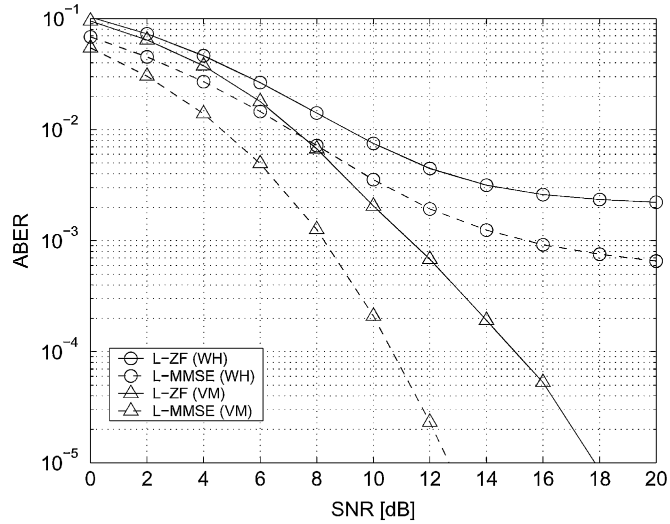

Fig. 1. ABER versus SNR (CP-based downlink, underloaded system with $J = 10$ users, linear receiving structures).

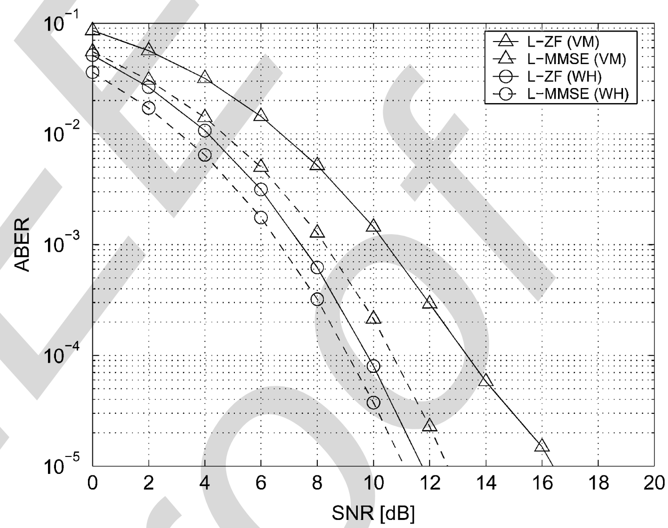

Fig. 2. ABER versus SNR (ZP-based downlink, underloaded system with $J = 10$ users, linear receiving structures).

$\overline{\mathbf{C}}$ in overloaded environments, which is a necessary condition for the existence of WL-ZF solutions in CP-based systems. In addition, since the code matrix given by (23) does not satisfy condition $\overline{\text{D}}_{\text{cp}}$, the curves of both the "WL-ZF (CWH)" and "WL-MMSE (CWH)" detectors exhibit an unacceptable floor for moderate-to-high values of the SNR. In contrast, it can be seen from the same figure that the proposed RVM spreading vectors (26), which ensure the existence of universal WL-ZF solutions in both CP- and ZP-based overloaded systems, allow the "WL-ZF (RVM)" and "WL-MMSE (RVM)" receivers to work very well. Fig. 4 reports the ABER curves of the receivers under comparison for a ZP-based system. We recall that, in this case, the full-column rank property of $\overline{\mathbf{C}}$ is a necessary and sufficient condition for the existence of WL-ZF solutions. Indeed, besides corroborating the uselessness of the "WL-ZF (WH)," "WL-MMSE (WH)," "WL-ZF (VM)," and "WL-MMSE (VM)" receivers in the considered overloaded setting, results of Fig. 4 confirm that both the proposed CWH and RVM code vectors

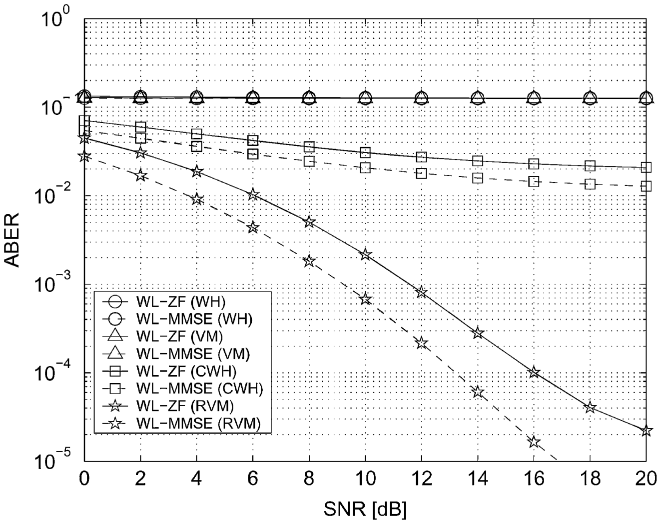

Fig. 3. ABER versus SNR (CP-based downlink, overloaded system with $J = 20$ users, WL receiving structures).

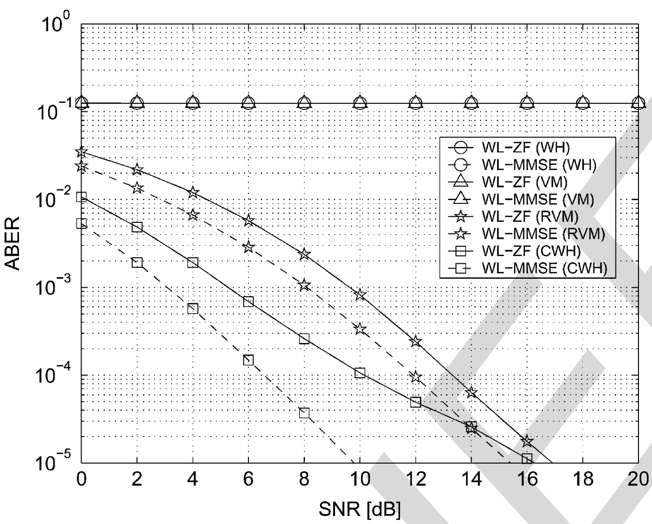

Fig. 4. ABER versus SNR (ZP-based downlink, overloaded system with $J = 20$ users, WL receiving structures).

ensure the existence of universal WL-ZF solutions, by showing that the curves of the "WL-ZF (CWH)," "WL-MMSE (CWH)," "WL-ZF (RVM)," and "WL-MMSE (RVM)" rapidly fall away as the SNR goes up. In particular, as already evidenced in the linear case, due to noise amplification effects, the "WL-ZF (CWH)" and "WL-MMSE (CWH)" detectors perform better than the corresponding counterparts "WL-ZF (RVM)" and "WL-MMSE (RVM)", expecially for low SNR values, by guaranteeing a significant saving in transmitter power, for a given value of the ABER.

2) ABER Versus Number of Users: In the second group of experiments, the performances of the considered receivers were studied as a function of the number J of active users, by setting SNR $= 10$ dB. As previously done, we investigated the performances of linear and WL receivers separately.

Figs. 5 and 6 report the performances of the L-ZF and L-MMSE detectors, when they are employed in both CP- and ZP-based underloaded systems, which use either WH or VM

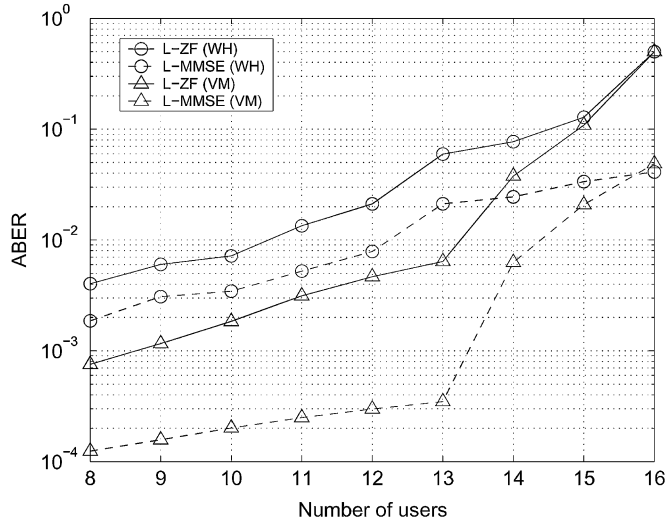

Fig 5. ABER versus number J of users (CP-based downlink, SNR $= 10$ dB, linear receiving structures).

spreading sequences (the observation made in footnote 7 still applies to this case). With reference to a CP-based system, results of Fig. 5 shows that, as long as the number of users is less than the threshold $N - L = 13$ (see Remarks 3 and 4), the "L-ZF (VM)" and "L-MMSE (VM)" detectors significantly outperform their "L-ZF (WH)" and "L-MMSE (WH)" corresponding counterparts. However, as soon as the number of active users gets over $J = 13$, in which case universal perfect symbol recovery in the absence of noise cannot be guaranteed, the performances of both the "L-ZF (VM)" and "L-MMSE (VM)" detectors rapidly deteriorate as the system load grows, by approaching the curves of the "L-ZF (WH)" and "L-MMSE (WH)" receivers. On the other hand, it can be seen from Fig. 6 that, for a ZP-based downlink, wherein the linear independence of the code vectors is a sufficient and necessary condition for assuring up to N users the existence of universal ZF solutions, all the receivers under comparison enable to achieve a greater system capacity than a CP-based system. In particular, according with the results of Fig. 2, the WH spreading sequences allow both the "L-ZF (WH)" and "L-MMSE (WH)" detectors to outperform their "L-ZF (VM)" and "L-MMSE (VM)" counterparts, respectively, for all the considered values of J. It is worthwhile to note that, with $J = N = 16$ users, the "L-MMSE (WH)" detector is able to assure an ABER of about $5 \cdot 10^{-4}$ at its output, whereas the "L-MMSE (VM)" one exhibits competitive performances, i.e., less than $5 \cdot 10^{-4}$, only up to 15 users.

In the last two experiments, we investigated the performances of the WL-ZF and WL-MMSE detectors as a function of the number J of users, ranging from an underloaded $(J \leq N)$ to an overloaded $(J > N)$ system. For a CP-based downlink, it can be seen from Fig. 7 that, paying no attention to the uninteresting cases of WH and CWH spreading sequences, which do not guarantee channel-irrespective perfect symbol recovery in both underloaded and overloaded CP-based systems, the "WL-ZF (VM)" and "WL-MMSE (VM)" detectors perform comparably to the "WL-ZF (RVM)" and "WL-MMSE (RVM)" ones only for $J = 12$ active users. Beyond this value, while

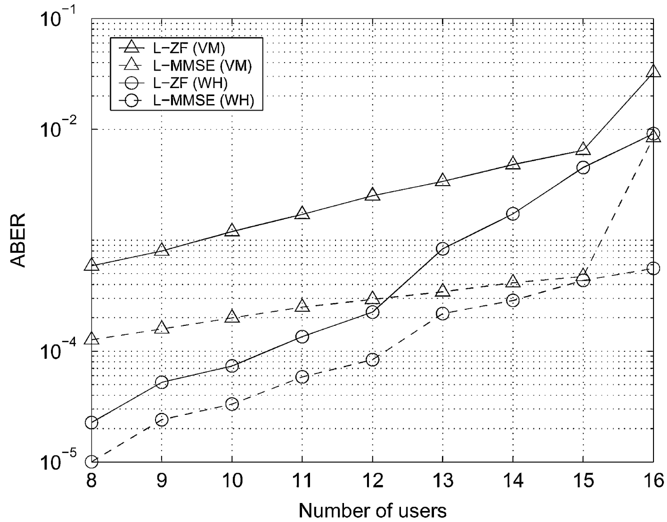

Fig. 6. ABER versus number J of users (ZP-based downlink, SNR $= 10$ dB, linear receiving structures).

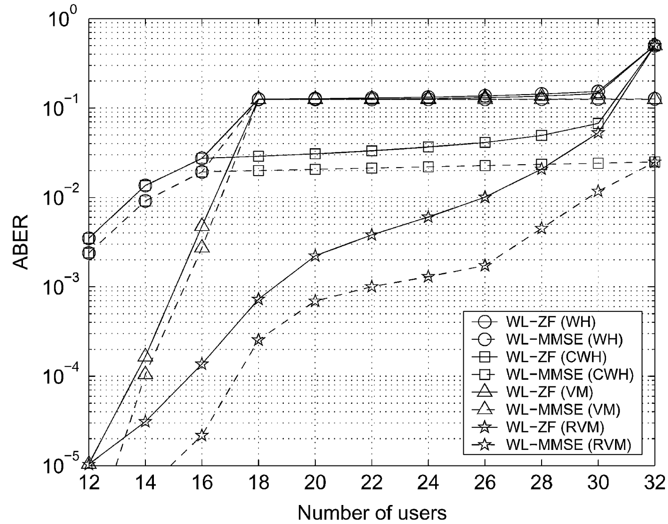

Fig. 7. ABER versus number J of users (CP-based downlink, SNR $= 10$ dB, WL receiving structures).

the performances of the "WL-ZF (VM)" and "WL-MMSE (VM)" receivers get worse very quickly, both the "WL-ZF (RVM)" and "WL-MMSE (RVM)" detectors still work satisfactorily up to $2(N-L) = 26$ users (see Remarks 9 and 11), by exhibiting ABER values less than or equal to 10^{-2} and $2 \cdot 10^{-3}$, respectively. Beyond the threshold $J = 26$, whereupon the existence of universal WL-ZF solutions cannot be ensured, the performances of the "WL-ZF (RVM)" and "WL-MMSE (RVM)" detectors rapidly worsen as J increases, and became comparable to those of the "WL-ZF (CWH)" and "WL-MMSE (CWH)" receivers. Finally, with reference to a ZP-based system, the curves depicted in Fig. 8 evidence that the performances of the "WL-ZF (WH)", "WL-MMSE (WH)", "WL-ZF (VM)" and "WL-MMSE (VM)" receivers are very poor when the system becomes overloaded. Furthermore, it is apparent that the proposed RVM and CWH code vectors allow both the WL-ZF and WL-MMSE receiver to manage a number of users which is significantly larger than the number of subcarriers. Remarkably, with $J = 2N = 32$ users, the "WL-MMSE (CWH)" detector is able to assure an ABER of $4 \cdot 10^{-4}$ at its output, whereas the ABER performance of the "WL-MMSE (VM)" is below 10^{-3} up to 30 users. On the basis of these experiments, we maintain that, among the different spreading techniques considered herein, the RVM code vectors turn out to be the best choice for both underloaded and overloaded CP-based systems, equipped with both linear and WL receiving structure, whereas the CWH spreading vectors allow both linear and WL detectors to exhibit the best performances in both underloaded and overloaded ZP-based systems.

VI. Conclusion

We tackled the problem of deriving mathematical conditions guaranteeing perfect symbol recovery in the absence of noise for either CP-based or ZP-based MC-CDMA downlink transmissions, which employ frequency-domain symbol-spreading. This issue is important also for the synthesis of MMSE receivers, since the performances of MMSE detectors strongly depend on the existence of the corresponding ZF solutions.

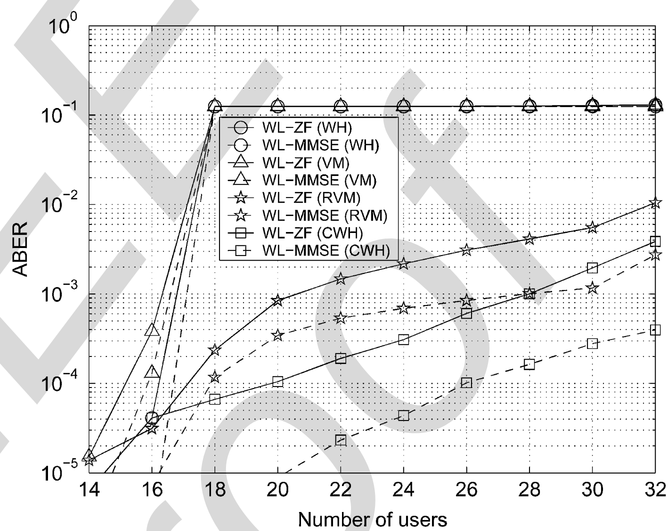

Fig. 8. ABER versus number J of users (ZP-based downlink, SNR $= 10$ dB, WL linear receiving structures).

The conditions derived in this paper are channel-independent and are expressed in terms of relatively simple system design constraints, regarding the maximum number of allowable users and their spreading sequences. Specifically, it was first shown that, similarly to a ZP-based MC-CDMA downlink and differently from CP-OFDM systems, L-ZF-MUD, which is confined only to underloaded systems and can be used when transmitted symbols are either proper or improper, can be guaranteed for a CP-based MC-CDMA downlink, even when the channel transfer function exhibits nulls on some used subcarriers. On the other hand, when the information-bearing symbols are improper, it was further shown that, for both CP-and ZP-based systems, WL-ZF-MUD allows one to successfully operate even in overloaded scenarios, by doubling the system capacity, regardless of the channel zero locations. However, such an increased throughput can be achieved as long as appropriate complex-valued spreading codes are used. Besides corroborating our theoretical findings, the performance analysis carried out by means of Monte Carlo simulations evidenced that the code designs provided herein allow L- and WL-MUD to achieve satisfactory performances, even when the parameters of the detectors are estimated through a finite data record. Finally, in this paper the channel impulse response was assumed to be exactly known at the receiving side; the interesting extension of blind subspace-based channel estimation is the topic of our current research and will be addressed in a forthcoming paper.

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