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Second-order statistics of one-sided CPM signals

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***Abstract*—This letter deals with second-order statistics (SOS) of continuous-phase modulated (CPM) signals. To overcome some mathematical inconsistencies emerging from the idealized assumption that the CPM signal evolves from $t = -\infty$, we consider a one-sided model for the signal, which starts from $t = 0$, noting also that such a model emerges naturally when building practical SOS estimators. On the basis of such a model, we first evaluate the SOS of the pseudo-symbols, which arise when expressing a CPM signal in terms of its Laurent representation, as well as closed-form expressions of the cyclic autocorrelation and conjugate correlation functions of one-sided CPM signals.**

***Index Terms*—CPM signals, Laurent representation, SOS.**

I. Introduction

KNOWLEDGE of *second-order statistics* (SOS) of the received signal is required in the synthesis of receiving structures based on quadratic cost measures, like the minimum-mean square error (MMSE), minimum-output energy (MOE), or maximum-signal-to-noise-ratio (SNR) criteria. When a bandpass signal is represented in terms of its complex envelope $x(t)$, SOS characterization requires [1], [2] evaluation of both its statistical *autocorrelation function* (ACF) $R_{xx}(t,\tau) \triangleq \mathbb{E}[x(t)\, x^*(t-\tau)]$ and its *conjugate correlation function* (CCF) $R_{xx^*}(t,\tau) \triangleq \mathbb{E}[x(t)\, x(t-\tau)]$. Since many man-made signals exhibit SOS that are periodic or almost periodic functions of time, i.e., the signals obey a *cyclostationary* or *almost cyclostationary* [3] model, their ACF and CCF can be expanded in a Fourier series with respect to the variable t, whose coefficients are the *cyclic* ACF/CCF [3].

Continuous-phase modulated (CPM) *signals* [4], [5] are widely employed in wireless communication systems, due to their favorable spectral and constant-modulus properties, as well as their good error-probability performance. Modeling and evaluation of the SOS of CPM signals is complicated by the memory and non-linearity of the modulation process. To this aim, a useful tool is the *Laurent representation* [6], wherein a CPM signal is expanded as a linear superposition of pulse-amplitude modulated (PAM) signals. Based on this representation, an expression for the ACF and power spectrum of the complex envelope of the CPM signal has been derived in [6]: however, no discussion about the CCF and cyclostationarity properties was provided.

In [7], evaluation of cyclic SOS and higher-order statistics (HOS) of CPM signals (including ACF and CCF) has been carried out in the nonstochastic or *fraction-of-time* (FOT) probability framework [8]; however, a problem of convergence of infinite products has been solved by introducing an "undetermined constant" that can assume values ± 1. A careful analysis of the derivations in [7] reveals that such a constant stems from the assumption that the CPM signal starts from $t = -\infty$: however, in practice, CPM signals evolve starting from a finite time-epoch. Moreover, practical SOS estimators are built by evaluating suitable time averages of sampled signal data, taken starting from a particular time epoch, e.g., $t = 0$.

We show in this letter that the aforementioned problem of convergence, as well as the practical issues of SOS estimation, can be dealt with by modeling the CPM signal as a *one-sided* random process [9], i.e., as a process that starts from $t = 0$. In particular, on the basis of such model and exploiting the linearity of the Laurent representation, we evaluate closed-form expressions for the cyclic ACF and CCF of the CPM signal, which depend in their turn on the SOS of the *pseudo-symbols* [6] of the Laurent representation.

II. One-sided CPM signal model

The complex envelope $x(t)$ of a continuous-time CPM signal with baud-rate $1/T$ defined for $t \geq 0$ (one-sided model) can be obtained by straightforward modifications of the classical two-sided model (see, e.g., in [4], [5]) as follows

$$x(t) = \exp\left[j2\pi h \sum_{n=0}^{+\infty} a_n\, g(t-nT) \right] \tag{1}$$

where h is the *modulation index*, the symbol sequence $\{a_n\}_{n\geq 0}$ assumes values in the M-ary alphabet $A \triangleq \{\pm 1, \pm 3, \ldots, \pm(M-1)\}$, $g(t) \triangleq \int_0^t f(u)\, du$ is the *phase response*, and $f(t)$ is the *frequency response* satisfying the three conditions: $f(t) \equiv 0$ for each $t \notin [0, LT]$; $f(t) = f(LT-t)$; and $\int_0^{LT} f(u)\, du = g(LT) = 1/2$, with $L \in \mathbb{N}$.

Assuming that h is not an integer and $M = 2$ (binary alphabet), by straightforward modifications of the Laurent representation proposed in [6], it can be proven that $x(t)$ for $t \geq 0$ is a linear superposition of $Q \triangleq 2^{L-1}$ PAM waveforms[1]

$$x(t) = \sum_{q=0}^{Q-1} \sum_{n=0}^{+\infty} s_{q,n}\, c_q(t-nT) \tag{2}$$

[1]Eqs. (1) and (2) should be slightly modified when $0 \leq t < (L-1)T$, to account for the finite-length of the Laurent pulses: however, such a transient phenomenon is not relevant when evaluating infinite-time averages.

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$$R_{s_{q_1} s_{q_2}^*}(n,m) = \prod_{\ell=0}^{m-1} \cos[\pi h(1-\beta_{q_1,\ell})] \prod_{\ell=0}^{-m-1} \cos[\pi h(1-\beta_{q_2,\ell})] \prod_{\ell=(m)^+}^{\min\left[n, L+m-1-(m)^+\right]} \cos\left[\pi h\left(2-\beta_{q_1,\ell}-\beta_{q_2,\ell-m}\right)\right]$$
$$\times \prod_{\ell=L-m}^{\min(n-m,L-1)} \cos\left[\pi h\left(2-\beta_{q_2,\ell}\right)\right] \prod_{\ell=L+m}^{\min(n,L-1)} \cos\left[\pi h\left(2-\beta_{q_1,\ell}\right)\right] [\cos(2\pi h)]^{\max\left[0, n-L+1-(m)^+\right]} \quad (9)$$

$$R^+_{s_{q_1} s_{q_2}^*}(m) = \prod_{\ell=0}^{m-1} \cos[\pi h(1-\beta_{q_1,\ell})] \prod_{\ell=0}^{-m-1} \cos[\pi h(1-\beta_{q_2,\ell})] \prod_{\ell=(m)^+}^{L+m-1-(m)^+} \cos\left[\pi h\left(2-\beta_{q_1,\ell}-\beta_{q_2,\ell-m}\right)\right]$$
$$\times \prod_{\ell=L-m}^{L-1} \cos\left[\pi h\left(2-\beta_{q_2,\ell}\right)\right] \prod_{\ell=L+m}^{L-1} \cos\left[\pi h\left(2-\beta_{q_1,\ell}\right)\right] [\cos(2\pi h)]^{-L+1-(m)^+} \quad (11)$$

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where the following *non-linear* functions of $\{a_n\}_{n\geq 0}$

$$s_{q,n} = \exp\left[j\pi h\left(\sum_{\ell=0}^{n} a_\ell - \sum_{\ell=0}^{\min(n,L-1)} a_{n-\ell}\beta_{q,\ell}\right)\right] \quad (3)$$

are the *pseudo-symbols*, with $n \geq 0$, $\beta_{q,\ell} \in \{0,1\}$ is the ℓth bit of the radix-2 representation of $q \in \{0,1,\ldots,Q-1\}$, i.e., $q = \sum_{\ell=1}^{L-1} 2^{\ell-1}\beta_{q,\ell}$ (with $\beta_{q,0}=0$), for $\ell \in \{1,2,\ldots,L-1\}$, and $c_q(t)$ is a real-valued pulse (see [6] for its expression). The Laurent representation can be extended to multilevel CPM signaling [10] and integer modulation indexes [11].

III. SOS of one-sided CPM signals

The time-averaged ACF of a two-sided CPM signal has been evaluated in [6, eq. (28)] in terms of its Laurent representation. The corresponding statistical ACF for the one-sided model can be written, for $t \geq (\tau)^+$, with $(\tau)^+ \triangleq \max(\tau,0)$, as

$$R_{xx}(t,\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{n=0}^{+\infty} \sum_{m=-\infty}^{n} R_{s_{q_1} s_{q_2}}(n,m) \times c_{q_1}(t-nT)\, c_{q_2}(t-nT+mT-\tau) \quad (4)$$

where $R_{s_{q_1} s_{q_2}}(n,m) \triangleq \mathbb{E}\left[s_{q_1,n} s^*_{q_2,n-m}\right]$, for $n \geq (m)^+$, with $(m)^+ \triangleq \max(m,0)$, is the cross-correlation function of the pseudo-symbols. Evaluation of the CCF for the two-sided CPM signal has not been carried out in [6]. It can be shown that, for the one-sided model, one has

$$R_{xx^*}(t,\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{n=0}^{+\infty} \sum_{m=-\infty}^{n} R_{s_{q_1} s^*_{q_2}}(n,m) \times c_{q_1}(t-nT)\, c_{q_2}(t-nT+mT-\tau) \quad (5)$$

where $R_{s_{q_1} s^*_{q_2}}(n,m) \triangleq \mathbb{E}\left[s_{q_1,n} s_{q_2,n-m}\right]$, for $n \geq (m)^+$, is the conjugate cross-correlation function of $\{s_{q,n}\}_{n\geq 0}$.

Both (4) and (5) depend on the SOS $R_{s_{q_1} s_{q_2}}(n,m)$ and $R_{s_{q_1} s^*_{q_2}}(n,m)$ of the pseudo-symbols, which are evaluated in the following.[2] Starting from (3), it can be proven that, for $n \geq (m)^+$, one has

$$R_{s_{q_1} s_{q_2}}(n,m) = [\cos(\pi h)]^{\Delta^+_{q_1 q_2}(n,m)} \quad (6)$$

where, for any $m \in \mathbb{Z}$, $\Delta^+_{q_1 q_2}(n,m) \triangleq \Delta_{q_1 q_2}(m) + \widetilde{\Delta}_{q_1 q_2}(n,m)$, where $\Delta_{q_1 q_2}(m)$ is an integer, whose explicit expression is given in [6, eq. (26)], whereas

$$\widetilde{\Delta}_{q_1 q_2}(n,m) = \sum_{\ell=n+1}^{\min(L+m-1,L-1)} 2\beta_{q_2,\ell-m}\,\beta_{q_1,\ell} - \sum_{\ell=n+1}^{L-1} \beta_{q_1,\ell} - \sum_{\ell=n-m+1}^{L-1} \beta_{q_2,\ell} \quad (7)$$

is a correction term, which vanishes for $n \geq L-1+(m)^+$; in the latter case, the cross-correlation function for the one-sided model turns out to be the same of that for the two-sided model after a small transient and, moreover, it does not depend on n, i.e., one has, for $n \geq L-1+(m)^+$,

$$R_{s_{q_1} s_{q_2}}(n,m) = R_{s_{q_1} s_{q_2}}(m) = [\cos(\pi h)]^{\Delta_{q_1 q_2}(m)}. \quad (8)$$

Moreover, starting again from (3), it can be inferred that, for $n \geq (m)^+$, the conjugate cross-correlation function of pseudo-symbols assumes the form[3] given in (9) at the top of this page. Note that, for $n \geq L-1+(m)^+$, (9) can be factorized as

$$R_{s_{q_1} s^*_{q_2}}(n,m) = R^+_{s_{q_1} s^*_{q_2}}(m)\,[\cos(2\pi h)]^n \quad (10)$$

where $R^+_{s_{q_1} s^*_{q_2}}(m)$ is given in (11) at the top of this page.

It is seen from (10) that for $h \neq \frac{1}{2}+k$, with $k \in \mathbb{Z}$, i.e., for $|\cos(2\pi h)| < 1$, the conjugate cross-correlation function of the pseudo-symbols vanishes as n increases; in this case, we will prove later that the one-sided CPM signal is asymptotically *circular* or *proper* (see [1], [2]). Instead, when $h = \frac{1}{2}+k$, with $k \in \mathbb{Z}$, i.e., $\cos(2\pi h) = -1$, the conjugate cross-correlation function of the pseudo-symbols does not vanish as n increases; in this case, the CPM signal exhibits asymptotically non-vanishing *noncircular* or *improper* [1], [2] features.

IV. Cyclic SOS of one-sided CPM signals

With reference to the one-sided CPM signal model, we observe [see (4) and (5)] that the signal exhibits in general time-varying SOS. Such time-varying features cannot be estimated in practice unless a structured model for time variations is assumed. When time variations in SOS are described by a periodic or almost periodic model for $t \geq 0$, they can be

[2] Cyclic SOS and HOS of CPM signals have been calculated in [7] in the nonstochastic FOT framework, without however giving explicit expressions for the SOS of the pseudo-symbols.

[3] When the lower-limit of a sum [product] is larger than its upper-limit, the sum [product] is conventionally equal to zero [one].

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$$R_{xx}^{\alpha}(\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \langle \sum_{n=(m)^+}^{L-2+(m)^+} R_{s_{q_1}s_{q_2}}(n,m)\, p_{q_1q_2}(t-nT,\tau-mT)\, e^{-j2\pi\alpha t}\rangle_+ + \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R_{s_{q_1}s_{q_2}}(m) \langle \sum_{n=L-1+(m)^+}^{+\infty} p_{q_1q_2}(t-nT,\tau-mT)\, e^{-j2\pi\alpha t}\rangle_+ \quad (16)$$

$$R_{xx^*}^{\alpha}(\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \langle \sum_{n=(m)^+}^{L-2+(m)^+} R_{s_{q_1}s^*_{q_2}}(n,m)\, p_{q_1q_2}(t-nT,\tau-mT)\, e^{-j2\pi\alpha t}\rangle_+ + \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R^+_{s_{q_1}s^*_{q_2}}(m) \langle \sum_{n=L-1+(m)^+}^{+\infty} [\cos(2\pi h)]^n\, p_{q_1q_2}(t-nT,\tau-mT)\, e^{-j2\pi\alpha t}\rangle_+ \quad (22)$$

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conveniently measured and estimated by defining the cyclic ACF at the cycle frequency $\alpha \in \mathbb{R}$ as

$$R_{xx}^{\alpha}(\tau) = \langle R_{xx}(t,\tau)\, e^{-j2\pi\alpha t}\rangle_+ \quad (12)$$

where $\langle f(t,\tau)\rangle_+ \triangleq \lim_{Z\to+\infty} \frac{1}{Z}\int_{(\tau)^+}^{Z} f(t,\tau)\, dt$ denotes the one-sided time average operator. An estimator of (12) is the finite time-average of $x(t)\, x^*(t-\tau)\, e^{-j2\pi\alpha t}$:

$$\widehat{R}_{xx}^{\alpha}(\tau) = \frac{1}{Z}\int_{(\tau)^+}^{Z} x(t)x^*(t-\tau)\, e^{-j2\pi\alpha t} dt\,. \quad (13)$$

It is clear that $\widehat{R}_{xx}^{\alpha}(\tau)$ is an asymptotically (for $Z \to +\infty$) unbiased estimator of $R_{xx}^{\alpha}(\tau)$; under mild conditions (see [12], [13]), it can be proven that it is also a consistent estimator.

To obtain the theoretical expression of $R_{xx}^{\alpha}(\tau)$, (4) must be substituted in (12). It is convenient first to rewrite (4) as

$$R_{xx}(t,\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \sum_{n=(m)^+}^{+\infty} R_{s_{q_1}s_{q_2}}(n,m) \times p_{q_1q_2}(t-nT,\tau-mT) \quad (14)$$

where $p_{q_1q_2}(t,\tau) \triangleq c_{q_1}(t)c_{q_2}(t-\tau)$. Thus, one has

$$R_{xx}^{\alpha}(\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \langle \sum_{n=(m)^+}^{+\infty} R_{s_{q_1}s_{q_2}}(n,m) \times p_{q_1q_2}(t-nT,\tau-mT)\, e^{-j2\pi\alpha t}\rangle_+\,. \quad (15)$$

Recalling the transient analysis of $R_{s_{q_1}s_{q_2}}(n,m)$ discussed with reference to (6)–(8), the time average in (15) can be decomposed as shown in (16) at the top of this page. The first time average in (16) is zero due to the finite duration of the signal involved. With reference to the second term, the two-sided version of the sum over n is clearly periodic in t of period T, thus it can be expanded as

$$\sum_{n=-\infty}^{+\infty} p_{q_1q_2}(t-nT,\tau-mT) = \sum_{k=-\infty}^{+\infty} X_k\, e^{j2\pi\frac{k}{T}t} \quad (17)$$

where $\{X_k\}_{k\in\mathbb{Z}}$ are the Fourier series coefficients, given by $X_k \triangleq \frac{1}{T}P_{q_1q_2}(f,\tau-mT)\big|_{f=k/T}$, where

$$P_{q_1q_2}(f,\tau) \triangleq \int_{-\infty}^{+\infty} p_{q_1q_2}(t,\tau)\, e^{-j2\pi ft}dt = \int_{-\infty}^{+\infty} C_{q_1}(\lambda)\, C^*_{q_2}(\lambda-f)\, e^{j2\pi(\lambda-f)\tau}d\lambda \quad (18)$$

is the Fourier transform of $p_{q_1q_2}(t,\tau)$ with respect to t, and the second expression arises by virtue of Parseval identity, with $C_q(f)$ denoting the Fourier transform of $c_q(t)$. Since (17) holds for any t, it holds *a fortiori* for the values of t involved in the one-sided sum over n in (16); by substituting (17) in (16), one has

$$R_{xx}^{\alpha}(\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R_{s_{q_1}s_{q_2}}(m) \sum_{k=-\infty}^{+\infty} X_k \langle e^{j2\pi(\frac{k}{T}-\alpha)t}\rangle_+\,. \quad (19)$$

Observing that $\langle e^{j2\pi(\frac{k}{T}-\alpha)t}\rangle_+ = \delta_{\alpha-k/T}$, with δ_k denoting the Kronecker delta, (19) shows that the CPM signal exhibits, in general, *wide-sense cyclostationarity* [14] with *cycle frequencies* $\alpha = \frac{k}{T}$, with $k \in \mathbb{Z}$, thus one has

$$R_{xx}^{\frac{k}{T}}(\tau) = \frac{1}{T}\sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R_{s_{q_1}s_{q_2}}(m)\, P_{q_1q_2}\left(\frac{k}{T},\tau-mT\right) \quad (20)$$

with $R_{s_{q_1}s_{q_2}}(m)$ and $P_{q_1q_2}(f,\tau)$ given by (8) and (18), respectively.

Let us consider now the CCF given by (5), which can be conveniently rewritten as

$$R_{xx^*}(t,\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \sum_{n=(m)^+}^{+\infty} R_{s_{q_1}s^*_{q_2}}(n,m) \times p_{q_1q_2}(t-nT,\tau-mT)\,. \quad (21)$$

Let us define the cyclic CCF $R_{xx^*}^{\alpha}(\tau) \triangleq \langle R_{xx^*}(t,\tau)\, e^{-j2\pi\alpha t}\rangle$, on the basis of (9) and (10), one obtains (22) at the top of this page. Similarly to (16), it can be proven that the first time average in (22) goes to zero due to the finite duration of the signal involved. With reference to the second term, two distinct cases must be discussed, according to the value of h.

Let us first consider the case where $h \neq \frac{1}{2} + k$, which implies that $|\cos(2\pi h)| < 1$; one has

$$\langle \sum_{n=L-1+(m)^+}^{+\infty} [\cos(2\pi h)]^n p_{q_1q_2}(t-nT,\tau-mT)\, e^{-j2\pi\alpha t}\rangle_+ = \lim_{Z\to+\infty}\frac{1}{Z}\sum_{n=L-1+(m)^+}^{+\infty}[\cos(2\pi h)]^n \times \int_{-\infty}^{+\infty} 1_{[(\tau)^+,Z]}(t)\, p_{q_1q_2}(t-nT,\tau-mT)\, e^{-j2\pi\alpha t}dt \quad (23)$$

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$$\left| \frac{1}{Z} \sum_{n=L-1+(m)^+}^{+\infty} [\cos(2\pi h)]^n \, [Z-(\tau)^+] \int_{-\infty}^{+\infty} \mathrm{sinc}\left\{[Z-(\tau)^+]f\right\} e^{-j2\pi f z(\tau)} P^*_{q_1 q_2}(f-\alpha, \tau - mT)\, e^{j2\pi(f-\alpha)nT} df \right| \\ \leq \frac{1}{1-\cos(2\pi h)} \int_{-\infty}^{+\infty} \left|\mathrm{sinc}\left\{[Z-(\tau)^+]f\right\}\right| \left|P_{q_1 q_2}(f-\alpha, \tau-mT)\right| df \quad (25)$$

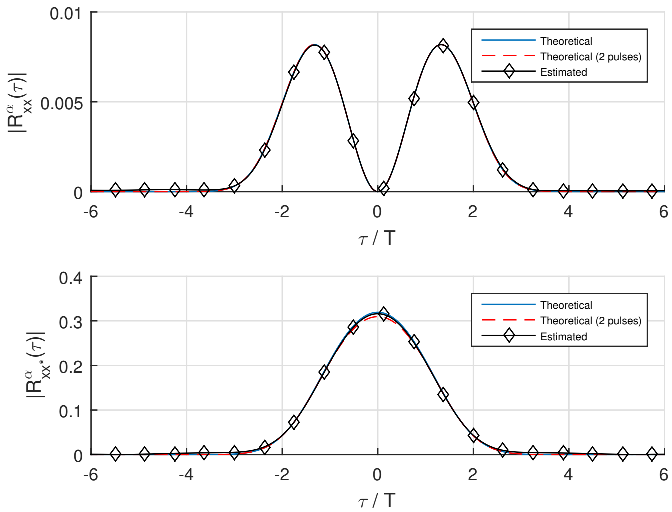

Fig. 1. Magnitude of the cyclic ACF (upper) at $\alpha = 1/T$ and cyclic CCF (lower) at $\alpha = 1/2T$ for a GMSK signal with $h = 0.5$, $L = 4$, and $BT = 0.25$.

where $1_{[a,b]}(t) = 1$ for $t \in (a, b)$ and zero otherwise. The last integral in (23) represents a scalar product in $L^2(\mathbb{R})$, thus, resorting to Parseval identity and applying straightforward properties of the Fourier transform, it can be expressed as

$$\int_{-\infty}^{+\infty} 1_{[(\tau)^+, Z]}(t)\, [p_{q_1 q_2}(t-nT, \tau-mT)\, e^{j2\pi\alpha t}]^* dt \\ = [Z-(\tau)^+] \int_{-\infty}^{+\infty} \mathrm{sinc}\left\{[Z-(\tau)^+]f\right\} \\ \times e^{-j2\pi f z(\tau)} P^*_{q_1 q_2}(f-\alpha, \tau-mT)\, e^{j2\pi(f-\alpha)nT} df \quad (24)$$

where $z(\tau) \triangleq \frac{Z+(\tau)^+}{2}$. Thus, the inequality shown in (25) at the top of this page can be simply derived. The last integral in (25) goes to zero when $Z \to +\infty$ due to the behavior of the sinc function. Thus, when $h \neq \frac{1}{2} + k$, a CPM signal exhibits a zero CCF, hence it obeys a circular signal model.

Instead, let us consider the case where $h = \frac{1}{2} + k$, with $k \in \mathbb{Z}$, in which case $\cos(2\pi h) = -1$. It turns out that

$$R^{\alpha}_{xx^*}(\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R^+_{s_{q_1} s^*_{q_2}}(m) \\ \times \left\langle \sum_{n=L-1+(m)^+}^{+\infty} (-1)^n p_{q_1 q_2}(t-nT, \tau-mT)\, e^{-j2\pi\alpha t} \right\rangle_+ . \quad (26)$$

If we consider the two-sided version of the sum over n, we observe that it is periodic in t of period $2T$, thus it can be expanded as

$$\sum_{n=-\infty}^{+\infty} (-1)^n p_{q_1 q_2}(t-nT, \tau-mT) = \sum_{k=-\infty}^{+\infty} Y_k\, e^{j2\pi \frac{k}{2T} t} \quad (27)$$

where $\{Y_k\}_{k\in\mathbb{Z}}$ are the Fourier series coefficients, given by $Y_k \triangleq \frac{1}{2T} G_{q_1 q_2}(f, \tau-mT)\big|_{f=k/2T}$, where

$$G_{q_1 q_2}(f, \tau) \triangleq P_{q_1 q_1}(f, \tau) \left[1 - e^{-j2\pi f T}\right] . \quad (28)$$

In this case, reasoning similarly to (19), it can be proven that the CPM signal exhibits, in general, *conjugate wide-sense cyclostationarity* [14] with *cycle frequencies* $\alpha = \frac{k}{2T}$, with $k \in \mathbb{Z}$, thus one has

$$R^{\frac{k}{2T}}_{xx^*}(\tau) = \frac{1}{2T} \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R^+_{s_{q_1} s^*_{q_2}}(m) \\ \times G_{q_1 q_2}\left(\frac{k}{2T}, \tau - mT\right) . \quad (29)$$

However, taking into account (28), it turns out that

$$G_{q_1 q_2}\left(\frac{k}{2T}, \tau-mT\right) = \begin{cases} 2\, P_{q_1 q_2}\left(\frac{k}{2T}, \tau-mT\right), & k \text{ odd;} \\ 0, & k \text{ even;} \end{cases} \quad (30)$$

hence the cyclic CCF is nonzero only for $\alpha = \frac{1}{2T} + \frac{k}{T}$, with $k \in \mathbb{Z}$, and one has

$$R^{\frac{1}{2T}+\frac{k}{T}}_{xx^*}(\tau) = \frac{1}{T} \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R^+_{s_{q_1} s^*_{q_2}}(m) \\ \times P_{q_1 q_2}\left(\frac{k}{T} + \frac{1}{2T}, \tau - mT\right) \quad (31)$$

for all $k \in \mathbb{Z}$, with $R^+_{s_{q_1} s^*_{q_2}}(m)$ and $P_{q_1 q_2}(f, \tau)$ given by (11) and (18), respectively.

To validate our findings, we plot in Fig. 1 the cyclic ACF given by (20) and CCF given by (31) for a Gaussian Minimum-Shift Keying (GMSK) signal with $h = 0.5$, $L = 4$, and $BT = 0.25$, together with their estimates obtained over 2^{15} symbols, with 2^5 samples/symbol. We also report the approximate expressions of the cyclic ACF and CCF obtained by considering only the first two Laurent pulses, which contain a large portion of the signal energy in many cases of interest [15]. All the curves show a very good agreement between simulation and analytical results.

V. Conclusions

In this letter, by adopting a one-sided model for a CPM signal, we derived closed-form expressions for its cyclic SOS, in terms of the SOS of the pseudo-symbols of its Laurent representation. The obtained expressions can be useful to design receiving structures for CPM signals based on optimization of quadratic cost-functions.

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